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THE AIRPLANE AS AN OBJECT OF CONTROL

(Block Diagrams of the Equations of Perturbed Airplane Motion)

By V. S. Vedrov, G. L. Romanov, and V. N. Surina

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ABSTRACT

This work considers the presentation of equations for the perturbed motion of an airplane in the form of single-loop block diagrams. A brief analysis is given of the characteristics of the individual links and of their change with changing flight speed and altitude. A derivation of transfer functions for control with elevator, rudder, and aileron is presented, as well as simplified expressions for the transfer functions, depending on the frequency range, which correspond to breaking up the perturbed motion into simple types.

The representation of the equations of perturbed airplane motion in the form of single-loop block diagrams permits a simple and easy application of contemporary methods of control theory to the analysis of airplane motion, and also allows rapid formulation of simplified equations of motion and transfer functions applicable during control with the control elements.

INTRODUCTION

In the last 15 to 20 years the theory of automatic control has reached a very high degree of development. The methods of this theory permit one, on the basis of physical ideas, to analyze relatively simply and conveniently, the stability and nature of the transmission characteristics of dynamical systems. This refers in particular to systems which are represented in the form of single-loop diagrams consisting of simple links [3].

However, this progress of the general theory has hardly made contact with the theory of dynamic aircraft stability, either with autopilots or without them. Just as it was 20 years ago, stability calculations are conducted by classic methods through investigation of the roots of characteristic equations, which requires lengthy calculation and still does not allow the assignment of a physically simple and easily perceived representation of the process of control and steering of aircraft.

Only in the very recent past have a few works dedicated to these questions appeared. From the Soviet literature, the book by I. V. Ostoslavskii and V. S. Kalachev should be pointed out. Basically, this book applies frequency methods to the analysis of short-period longitudinal oscillations. Work by Spearman and some others (see list of references) presents the theory of transfer functions and frequency characteristics of aircraft for cases of rapid motions about the center of gravity.

However, all these works are either limited to the analysis of individual types of motion, or they are based on simplifying assumptions which are mostly unproven and occasionally wrong. But there does exist the possibility of a complete analysis of airplane motion which does not include any unfounded assumptions and encompasses all types of motion. It is possible to represent the equations of motion of an airplane in the form of a simple block diagram, in which it is easy to apply modern methods of control theory: circuit analysis, frequency methods, theory of compensating networks, etc. This report is devoted to those problems.

Any system of linear differential equations with constant coefficients can be represented in the form of a block diagram; however this diagram usually turns out to be multi-loop. It is difficult to apply the methods of

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control theory to such a diagram. On the other hand, any system can be represented as a single-loop diagram; however the links of such a system will be devoid of physical meaning. Therefore, in our work we attempted to present the system of equations of perturbed motion in such a form that it could be expressed by a single-loop diagram with links of the first or second order, and with signals having a simple physical meaning at the input and output of the links.

This work is not claimed to be a complete analysis of perturbed airplane motion through single-loop block diagrams. It is intended to point out the possibility of such a representation, and to show its advantages. Therefore, to illustrate the proposed method only airplanes with "conventional" characteristics are considered. In particular it is assumed that the airplane possesses sufficient longitudinal static stability, directional stability, etc.

In analyzing the perturbed motion of an airplane, we will not make any assumptions except those which are normally made in the theory of dynamic airplane stability [1, 2]. In particular we shall always assume that the equations of motion are linearized. Separate simplifying assumptions will always be pointed out in the appropriate places, and their use justified.

We note, however, that all the discussion and conclusions of this work are applicable only in the case when the initial unperturbed motion takes place in the symmetry plane of the airplane. In particular, in the initial motion

$$\gamma=\beta=\omega_x=\omega_y=0. \tag{1}$$

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SYMBOLS

t – time

p – Laplace operator; in this work $p = \frac{d}{dt}$

$W(p)$ – transfer function of a link or system of links

m – mass of the airplane

α – angle of attack at the center of gravity

β – angle of sideslip at the center of gravity

H – flight altitude

ψ, ϑ, γ – angles of yaw, pitch, and bank

θ – angle of inclination of the trajectory to the horizontal

V – velocity of the airplane's center of gravity

ω_x, ω_y, ω_z – projections of angular velocity along the airplane axes

$Y=\frac{1}{2}c_y\rho SV^2$ – lift force

$Q=\frac{1}{2}c_x\rho SV^2$ – drag

P – thrust

$X=Q-P$ – braking force (net component of all aerodynamic forces, including thrust, along the tangent to the trajectory)

$M_x=\frac{1}{2}m_x\rho SV^2l$, $M_y=\frac{1}{2}m_y\rho SV^2l$, $M_z=\frac{1}{2}m_z\rho SV^2l$ – components along the principal axes of the plane, of the moment of the aerodynamic forces* including the thrust, taken about the center of gravity.

* Coordinate axes x, y, z originate at the center of gravity and are directed along the principal axes of the airplane. Axis Ox–forward, axis Oz–to the right, axis Oy–upward, perpendicular to Ox and Oz.

$Z = \frac{1}{2} c_z \rho S V^2$ – Lateral component of aerodynamic forces along the z axis

l – wing span

b – chord length (average)

$\mu = \frac{2m}{\rho S b}$ – relative longitudinal density of the airplane

$\mu = \frac{2m}{\rho S l}$ – relative lateral density of the airplane

$\tau = \frac{2m}{\rho S V}$ – unit of time reduced to dimensionless form

$\bar{r}_x = \frac{1}{l}\sqrt{\frac{J_x}{m}}$, $\bar{r}_y = \frac{1}{l}\sqrt{\frac{J_y}{m}}$, $\bar{r}_z = \frac{1}{b}\sqrt{\frac{J_z}{m}}$ – dimensionless radii of gyration

ω – frequency of oscillations

ζ – relative coefficient of damping of a second order link

k – amplification factor* of a link or system of links.

The subscript indicates the derivative with respect to the parameter in the superscript. For example, $Z^\beta = \frac{\partial Z}{\partial \beta}$.

* Translator's Note: Many Americans would prefer to call k simply the "value of the transfer function at zero frequency." The term "amplification factor" is particularly misleading in cases where the input and output quantities have different dimensions and hence are not directly comparable.

$$W_1(p) = \frac{k_1}{T_1^2 p^2 + 2\zeta_1 T_1 p + 1},$$

in which $k_1 = \frac{1}{A_0}$ is the amplification factor* of link I;

$\zeta_1 = \frac{A_1}{2\sqrt{A_0}}$ is the relative damping coefficient;

$T_1 = \frac{1}{\sqrt{A_0}}$ is the time constant of the link; and

$\omega_1 = \frac{1}{T_1}$ is the reference frequency of the link.

It is easy to see that k_1 is directly proportional to the square of the time unit τ^2, that is, inversely proportional to the squares of the flight velocity and air density.

The reference frequency of link I increases, other conditions being equal, in direct proportion to the flight velocity and decreases with decrease of the density ρ. The relative damping coefficient ζ_1 does not depend directly on the speed, and decreases almost proportionally to $\sqrt{\rho}$.

The expanded expression for the transfer function of link II has the following form:

$$W_2(p) = \frac{1}{p^2 + \left[\left(c_x + \frac{M}{2} c_x^M\right)\frac{S\rho V}{m} - \frac{P^V}{m} - \frac{g}{m}\sin\theta\right]p + \left[\left(c_y + \frac{M}{2} c_y^M\right)\cos\theta - \left(c_x + \frac{M}{2} c_x^M\right)\sin\theta\right]\frac{S\rho}{m} g}.$$

First of all it is important to note that the reference frequency of link II

$$\omega_2 = \sqrt{\left[\left(c_y + \frac{M}{2} c_y^M\right)\cos\theta - \left(c_x + \frac{M}{2} c_x^M\right)\sin\theta\right]\frac{S\rho}{m} g}$$

does not depend directly on the velocity. It can vary only with the Mach number M. Therefore, the frequency ratio of links I and II will increase with increase of flight speed. The same can be said of the amplification factor $k_2 = \frac{1}{\omega_2^2}$.

The frequency ω_2 decreases proportionally to the decrease of $\sqrt{\rho}$, that is, it decreases with the increase in altitude, and the amplification factor k_2 increases with the decrease of ρ, in inverse proportion.

At some sufficiently large angle θ we can have the equation

$$\operatorname{tg}\theta = \frac{c_y + \frac{M}{2} c_y^M}{c_x + \frac{M}{2} c_x^M}$$

and, therefore, the decrease of ω_2 down to zero; with the further increase of θ, link II becomes unstable. The amplification factor k_2 will increase to $k_2 = \infty\ (\omega_2 = 0)$ with the increase of θ. Therefore, the expression of the transfer function in terms of k, T and ζ, as accepted in control theory is not always convenient, because in airplane dynamics it is frequently necessary to deal with unstable links and with the consequent difficulties of interpretation of indeterminate quantities containing 0 and ∞.

In particular, the relative coefficient of damping

* See translator's note, p. 3.

$$\zeta = \frac{\left(c_x + \frac{M}{2} c_x^M\right) \frac{S\rho V}{m} - \frac{P^V}{m} - \frac{g}{V} \sin\theta}{2\sqrt{\left[\left(c_y + \frac{M}{2} c_y^M\right)\cos\theta - \left(c_x + \frac{M}{2} c_x^M\right)\sin\theta\right]\frac{S\rho}{m} g}}$$

depends on V, ρ and θ in a very complex way.

In this case the damping of link II can be evaluated more simply through the damping coefficient of the natural motion of link II, which is equal to half the coefficient of p and time, during which the natural oscillation decreases by the factor $e \approx 2.718$.

The damping coefficient of link II for the aerodynamic data assumed for the calculation has following magnitude:

for the altitude H = 5,000 meters

$$\left(0{,}022 + \frac{0{,}69}{2} 0{,}019\right) \frac{0{,}0751 \cdot 222}{33{,}7} - \frac{9{,}81}{222} \sin\theta = 0{,}0142 - 0{,}0441 \sin\theta,$$

for the altitude H = 12,000 meters

$$\left(0{,}032 + \frac{0{,}75}{2} 0{,}081\right) \frac{0{,}0317 \cdot 222}{33{,}7} - \frac{9{,}81}{222} \sin\theta = 0{,}0134 - 0{,}0441 \sin\theta.$$

Thus it is obvious that with increasing angle of climb, the damping of link II decreases, and at $\theta = 18° - 19°$ link II becomes unstable, the natural motion being an increasing oscillation. This explains to a great degree the stability deterioration during a climb [1, 2].

During horizontal flight $\theta = 0$ and the damping of link II is directly proportional to flight speed and air density ρ. During a climb or descent, there appears in the damping coefficient a third term, which decreases the damping during ascent and increases it during descent, in both cases in inverse proportion to the flight speed.

The second term $\frac{P^Y}{m}$ for subsonic and trans-sonic planes with turbojet engines is small and unimportant. However, with increase of speed, the derivative of thrust $P^V = \frac{\partial P}{\partial V}$ of jet engines increases considerably; therefore, the damping of link II will decrease. It is conceivable that at high flight speeds this link can become unstable, or the stability will be so low that an artifical damping of the long period motions will become necessary, for example, by an automatic device which affects the thrust by an amount depending on the speed change during the oscillations.

The transfer function of link III has the following form:

$$W_3(p) = -\left[c_x^\alpha \frac{S}{m} \frac{\rho V^2}{2} p + (c_y^\alpha \cos\theta - c_x^\alpha \sin\theta) \frac{S\rho V}{2m} g\right].$$

The expression $W_3(p)$ can be conveniently represented in the following form:

$$W_3(p) = -k_3 [T_3 p + 1],$$

where

$$k_3 = (c_y^\alpha \cos\theta - c_x^\alpha \sin\theta) \frac{S\rho V}{2m} g = (c_y^\alpha \cos\theta - c_x^\alpha \sin\theta) \frac{g}{\epsilon}$$

is the amplification factor of link III;

$$T_3 = \frac{1}{\omega_3} = \frac{c_x^\alpha}{c_y^\alpha \cos\theta - c_x^\alpha \sin\theta} \frac{V}{g}$$

is the time constant of this link.

It is easy to see that the time constant T_3 of the link increases in direct proportion to flight speed. During horizontal flight $\theta = 0$ and therefore

$$T_3 = \frac{c_x^\alpha}{c_y^\alpha} \frac{V}{g} = \frac{\partial c_x}{\partial c_y} \frac{V}{g},$$

that is, T_3 is determined in this case by the tangent of the angle formed by the tangent line to the polar curve* for the plane (at M = const). With increasing angle θ during a climb, the denominator begins to decrease, and at a very large angle $\theta = \arc\tan \dfrac{c_y^\alpha}{c_x^\alpha}$ it becomes zero; at the same time, the amplification factor k_3 decreases in the same proportion. In order to see clearer the changes in characteristics of link III, let us express its transfer function in a different form:

$$W_3(p) = k_3 T_3 (p + \omega_3);$$

here the quantity $k_3 T_3 = c_x^\alpha \dfrac{S}{m} \dfrac{\rho V^2}{2} = \dfrac{X^\alpha}{m}$ does not depend on the angle θ. The reference frequency of the link

$$\omega_3 = \frac{1}{T_3} = \frac{c_y^\alpha \cos\theta - c_x^\alpha \sin\theta}{c_x^\alpha} \frac{g}{V}$$

is inversely proportional to flight speed. During a climb with a very large angle $\theta = \arc\tan \dfrac{c_y^\alpha}{c_x^\alpha}$ the reference frequency becomes zero; at this angle of ascent link III becomes a purely differentiating link of the first order with an amplification factor of $-\dfrac{X^\alpha}{m}$, and with further increase of angle θ link III becomes unstable.

Because of the decrease of reference frequency of link III with increase in velocity, its frequency characteristics are displaced towards the low-frequency side.

In the general case link IV is a differentiating link of the second order

$$W_4(p) = p^2 \frac{\operatorname{tg}\theta}{V} - \left(\frac{Y^V}{mV\cos\theta} + \frac{M_z^{\omega_z}}{J_z} \frac{\operatorname{tg}\theta}{V} \right) p + \frac{M_z^V}{J_z} =$$
$$= p^2 \frac{\operatorname{tg}\theta}{V} - \left\{ \left[c_y + \frac{M}{2} c_y^M - \left(c_x + \frac{M}{2} c_x^M \right) \operatorname{tg}\theta \right] \frac{S\rho}{m} + m_z^{\omega_z} \frac{\rho S}{2J_z} \operatorname{tg}\theta \right\} p +$$
$$+ \frac{M}{2} m_z^M \frac{S\rho V}{J_z} + m_z^{\omega_z} \left[c_y + \frac{M}{2} c_y^M - \left(c_x + \frac{M}{2} c_x^M \right) \operatorname{tg}\theta \right] \frac{S^2\rho^2 b^2 V}{2mJ_z}.$$

At $\theta = 0$ this link becomes a link of first order. It is easy to see that reference frequency of such a link increases in proportion to the flight velocity and decreases with decreasing air density.

Frequently the total aerodynamic moment depends only to a small degree on the Mach number M, that is, $m_z^M \approx 0$. In this case we obtain a quadratic expression which can easily be factored:

$$W_4(p) = \left[p - m_z^{\omega_z} \frac{b^2 \rho S V}{2J_z} \right] \left\{ p \frac{\operatorname{tg}\theta}{V} - \left[\left(c_y + \frac{M}{2} c_y^M \right) - \left(c_x + \frac{M}{2} c_x^M \right) \operatorname{tg}\theta \right] \frac{S\rho}{m} \right\}.$$

* Translator's Note: curve of c_y versus c_x.

Therefore, when $m_z^M = 0$, link IV represents the tandem combination of links of first order with transfer functions

$$W'_4(p) = p - m_z^{\bar{\omega}_z} \frac{b^2 \rho S}{2J_z} V,$$

$$W''_4(p) = p\frac{\operatorname{tg}\theta}{V} - \left[\left(c_y + \frac{M}{2}c_y^M\right) - \left(c_x + \frac{M}{2}c_x^M\right)\operatorname{tg}\theta\right]\frac{S\rho}{m}.$$

The first of these links is the same as link IV at $\theta = 0$, but without the multiplier

$$k'_4 = -\left(c_y + \frac{M}{2}c_y^M\right)\frac{S\rho}{m}.$$

At $\theta = 0$ the second link becomes a purely amplifying link with the amplification factor k_4. Conditionally we can consider this link as hiving infinite frequency; at $\theta \neq 0$ its frequency becomes finite and equals

$$\omega''_4 = \left|\left(c_y + \frac{M}{2}c_y^M\right)\frac{1}{\operatorname{tg}\theta} - \left(c_x + \frac{M}{2}c_x^M\right)\right|\frac{S\rho V}{m}.$$

With increase of angle θ this frequency decreases rapidly and can become equal to the frequency of the first link

$$\omega'_4 = -m_z^{\bar{\omega}_z}\frac{b^2\rho S}{2J_z}V.$$

At small angles θ the frequency ω_4'' is large and this second link can be regarded as amplifying. In other words, at small θ we can consider link IV as being identical to link IV at $\theta = 0$.

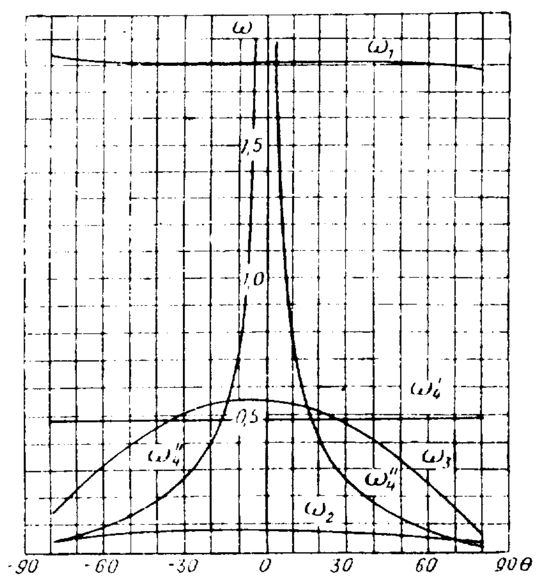

Fig. 4. The influence of the angle θ of inclination of the trajectory, upon the frequencies of link of the stabilization loop. Altitude 12 kilometers.

At large angles θ the frequency of link IV' remains unchanged, but the frequency of link IV" decreases rapidly. Figure 4 shows the change in frequency of all links of the stabilization loop with change of the angle θ, calculated for our example (H = 12 kilometers). Frequencies ω_1 and ω_4' practically do not change; frequencies ω_3 and ω_2 change insignificantly. A particularly large change takes place in the frequency ω_4'', which drops from an infinitely large value at $\theta = 0$ to 0 at $\theta = \pm 90°$.

In the case $M_Z^V \neq 0$ (when the Mach number M influences the coefficient m_Z) the analysis becomes more complex, but even in this case the same relations are still valid: link IV can usually be expressed in the form of two links of the first order connected in tandem; here at small θ the second link has a high frequency and we can practically consider link IV as a link of first order with the transfer function

$$W_4(p) \approx -\frac{Y^V - X^V \operatorname{tg}\theta}{mV}p + \frac{\bar{M}_z^V}{J_z}.$$

For large values of θ the reference frequency ω_4' will change negligibly, but the reference frequency ω_4'' will drop from infinity to 0.

Let us consider link V. When expanded the transfer function of this link will have the following form:

$$W_5(p)=\frac{Y^\alpha}{mV}p+\frac{\Delta(\alpha,V)}{V}=c_y^\alpha\frac{S\rho V}{2m}p+\left[c_y^\alpha\left(c_x+\frac{M}{2}c_x^M-c_P^V\right)-c_x^\alpha\left(c_y+\frac{M}{2}c_y^M\right)\right]\frac{S^2\rho^2V^2}{2m^2}=$$
$$=\frac{c_y^\alpha}{\tau}p+\left[c_y^\alpha\left(c_x+\frac{M}{2}c_x^M-c_P^V\right)-c_x^\alpha\left(c_y+\frac{M}{2}c_y^M\right)\right]\frac{2}{\tau^2},$$

where $c_P^V=\dfrac{\frac{\partial P}{\partial V}}{S\rho V}$ is a coefficient for the derivative of engine thrust with respect to speed.

It is convenient to represent this link in the form

$$W_5(p)=k_5[T_5p+1]=k_5T_5[p+\omega_5],$$

where

$$\omega_5=\frac{1}{T_5}=\left[\left(c_x+\frac{M}{2}c_x^M-c_P^V\right)-\frac{c_x^\alpha}{c_y^\alpha}\left(c_y+\frac{M}{2}c_y^M\right)\right]\frac{S\rho V}{m}$$

is the reference frequency of the link and

$$k_5=\left[c_y^\alpha\left(c_x+\frac{M}{2}c_x^M-c_P^V\right)-c_x^\alpha\left(c_y+\frac{M}{2}c_y^M\right)\right]\frac{S^2\rho^2V^2}{2m^2}$$

is the amplification factor of link V.

The natural frequency ω_5 increases proportionately to the speed and decreases proportionately to the air density. The amplification factor of link V increases proportionately to the square of the speed and decreases proportionately to the square of the air density.

In the formulas for ω_5 and k_5 the expression in parentheses can become zero at certain speeds and altitudes. If we disregard the influence of Mach number M on changes of the coefficents c_x and c_y and also neglect the coefficient c_P^V, then at a certain angle of attack the following equation will apply:

$$\frac{c_y}{c_x}=\frac{c_y^\alpha}{c_x^\alpha}=\frac{\partial c_y}{\partial c_x}.$$

In this case we will have $\Delta(\alpha, V) = 0$, that is, link V will become purely differentiating with amplification factor $\frac{Y^\alpha}{mV}$; at even larger angles of attack the quantity $\Delta(\alpha, V)$ will become negative, i. e., link V will become unstable. The quantity $\Delta(\alpha, V)$ becomes zero at an angle of attack which corresponds to the optimum regime.

For contemporary supersonic planes with jet engines the coefficient $c_P^V>0$ and reaches considerable magnitude. The influence of Mach number M on coefficients c_y and c_x, in general, cannot be neglected. Therefore, $\Delta(\alpha, V)$ usually becomes zero at angles of attack which are different from the angles corresponding to $\left(\frac{c_y}{c_x}\right)_{\max}$.

Let us consider link VI. Link VI is a differentiating link of second order. When expanded, the transfer function of this link takes the following form:

$$W_6(p)=p^2+\left[\left(\frac{c_y^\alpha}{2}+c_x+\frac{M}{2}c_x^M\right)\frac{S}{m}\rho V-\frac{g}{V}\sin\theta\right]p+$$
$$+\frac{1}{2}\left[c_y^\alpha\left(c_x+\frac{M}{2}c_x^M-c_P^V\right)-c_x^\alpha\left(c_y+\frac{M}{2}c_y^M\right)\right]\left(\frac{S\rho V}{m}\right)^2+$$
$$+\left[\left(c_y+\frac{M}{2}c_y^M\right)\cos\theta-\left(c_x+\frac{M}{2}c_x^M\right)\sin\theta\right]\frac{S\rho}{m}g.$$

The coefficient of p depends primarily on the first term and therefore increases proportionately to the flight speed, but with increase in altitude this term decreases proportionately to the air density ρ. In the first factor the term containing c_y^α is the most important.

Of the two terms most containing p, the first was already considered in link V. This is the reference frequency ω_5 with the multiplier $\frac{1}{c_y^\alpha}\frac{S\rho V}{m}$. The term increases proportionately to the square of flight speed. The second term does not depend on flight speed and can change only indirectly through the influence of Mach number M.

Usually link VI possesses more than critical damping and can be expressed in the form of links VI' and VI" of the first order connected in tandem:

$$W_6(p)=W_6'(p)\,W_6''(p)=(p+\omega_6')(p+\omega_6'').$$

In our example

at altitude 5,000 m, $W_6(p) = p^2 + 1{,}304p + 0.01932 = (p + 1.289)(p + 0.015)$;
at altitude 12,000 m, $W_6(p) = p^2 + 0.584p + 0.00826 = (p + 0.57)(p + 0.0145)$.

Figures 5 and 6 show the characteristics of first-order links VI' and VI", and also of the equivalent second-order link IV with transfer function $W_6(p)$.

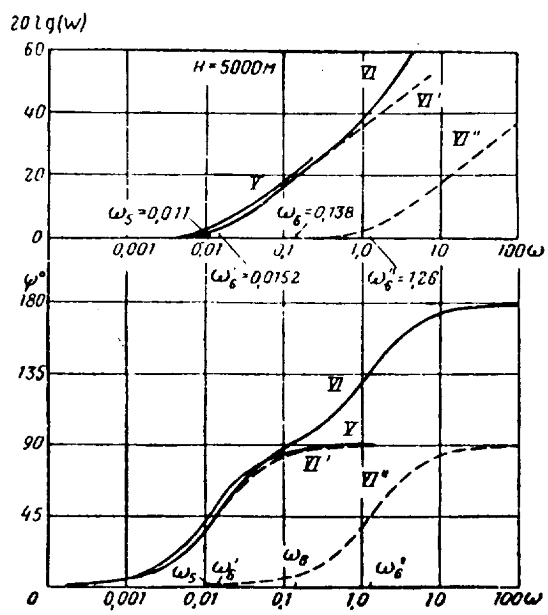

Fig. 5. Logarithmic frequency characteristics of control links for longitudinal motion. Altitude 5,000 m, velocity 800 kilometers per hour. Amplification factor: $k_5 = (0.0141/\mathrm{sec}^2)$ (−36.8 db); $k_6' = (1.289/\mathrm{sec})$ (2.2 db); k_6'' (0.015/sec)(−36.5 db); $k_6 = (0.01934/\mathrm{sec}^2)$ (− 34.3 db).

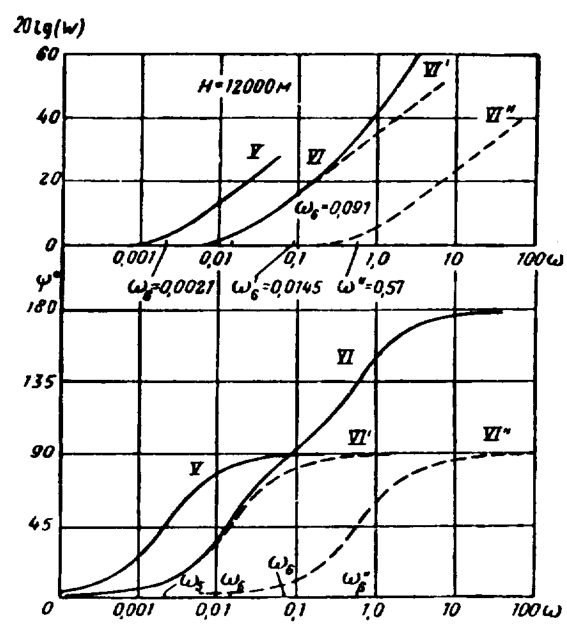

Fig. 6. Logarithmic frequency characteristics of control links for longitudinal motion. Altitude 12,000 m velocity 800 kilometers per hour. Amplification factor: $k_5 = (0.0121/\mathrm{sec}^2)$ (−58.3 db); $k_6' = (0.57/\mathrm{sec})$ (−4.9 db); $k_6'' = (0.0145/\mathrm{sec})$ (−36.6 db) $k_6 = (0.00855/\mathrm{sec}^2)$ (− 41.5 db).

Finally, let us look at link VII. In the denominator of its transfer function there is an expression which is equal to the transfer function of link V divided by $\underline{V}$.

In the numerator we can also separate out an operator, which describes link V, multiplied by $\underline{V}\cos\theta$. During horizontal flight at $\theta = 0$ this link VII becomes a simple amplifying link with the amplification factor $k_7 = 1$. At small values of angle θ we can approximately consider

$$W_7(p) = 1.$$

At large values of angle θ we have to carry out a detailed analysis of the transfer function of this link.

Figures 5 and 6 show the amplitude-frequency and phase-frequency characteristics of link V and VI, calculated for the case $\theta = 0$.

Section 3. Transfer Functions of Control Links. Simplification of Equations of Motion and Transfer Functions

The system of links I-IV form a closed loop for airplane stabilization. This loop can be conveniently opened after link IV. The transfer function of the opened system is

$$W(p) = -W_1(p)\,W_2(p)\,W_3(p)\,W_4(p). \tag{3.1}$$

Usually, in control theory [3] the transfer function of an open loop is prefixed by a plus sign. This is obtained because an error signal is supplied as an input into the first link. Because in our case the output signal of the last member is fed into the first member with the plus sign, and not a minus sign as is usually done in control theory and tracking systems, it is more convenient to prefix a minus sign to the transfer function. In this way, the usual stability criteria of general control theory are preserved.

The characteristic equation of the closed system will have following form:

$$\frac{1}{W_1(p)\,W_2(p)} - W_3(p)\,W_4(p) = \frac{1+W(p)}{W_1(p)\,W_2(p)} = 0.$$

It is easy to confirm by direct calculation that this equation agrees with the usual characteristic equation [1, 2]. At the same time, representation of the transfer function in the form of (3.1) allows us to apply the widely-known contemporary methods of control theory [3], particulary frequency methods, to the analysis of airplane stability. With this it is easy to see which link is the decisive one, and where it is necessary to introduce a correcting link.

In particular, the constant term a_4 of the characteristic equation equals

$$a_4 = \frac{1}{k_1k_2} - k_3k_4 = \frac{1}{k_1k_2}(1+k) = -\frac{\bar{M}_z^{\alpha}}{J_z}\frac{Y^V g}{mV} + \frac{\bar{M}_z^V Y^{\alpha} g}{J_z mV},$$

where $k = -k_1k_2k_3k_4 = W(0)$ is the amplification factor of the opened stabilization loop in the static regime ($p = 0$).

Expanding the equation for a_4, we obtain

$$a_4 = -\frac{M_z^{\alpha}}{J_z}\frac{g}{V}\frac{Y^V\cos\theta - X^V\sin\theta}{m} + \frac{M_z^V}{J_z}\frac{g}{V}\frac{Y^{\alpha}\cos\theta - X^{\alpha}\sin\theta}{m},$$

which corresponds to the usual expression [1].

From the condition of aperiodic stability

$$a_4 = \frac{1}{k_1k_2} - k_3k_4 > 0$$

it can be immediately seen that increase of the amplification factors k_1 and k_2, that is, decrease of the static stability and of the derivative of lift force with respect to speed, lowers the stability; conversely, an increase of the coefficients k_2 and k_4, that is, an increase of the derivative of lift force Y^{α} and increase of $\frac{M_z^Y}{J_z}$ (as a result of compressibility), acts in a stabilizing manner.

The equations of motion of the airplane can be expressed in an abbreviated form as follows:

$$\left.\begin{aligned} \Delta\alpha &= W_1(p)\left[W_4(p)\,\Delta V + \frac{M_z^{\delta}}{J_z}\,\Delta\delta\right], \\ \Delta V &= W_2(p)\,W_3(p)\,\Delta\alpha. \end{aligned}\right\} \qquad (3.2)$$

Hence

$$\left.\begin{aligned} \Delta\alpha &= \frac{W_1(p)}{1+W(p)}\,\frac{M_z^{\delta}}{J_z}\,\Delta\delta, \\ \Delta V &= \frac{W_1(p)\,W_2(p)\,W_3(p)}{1+W(p)}\,\frac{M_z^{\delta}}{J_z}\,\Delta\delta, \\ \Delta\vartheta &= W_2(p)\,W_6(p)\,\Delta\alpha = \frac{W_1(p)\,W_2(p)\,W_6(p)}{1+W(p)}\,\frac{M_z^{\delta}}{J_z}\,\Delta\delta, \\ \Delta\theta &= W_2(p)\,W_5(p)\,\Delta\alpha = \frac{W_1(p)\,W_2(p)\,W_5(p)}{1+W(p)}\,\frac{M_z^{\delta}}{J_z}\,\Delta\delta, \\ \Delta H &= \frac{1}{p}\,W_7(p)\,\Delta\theta = \frac{W_1(p)\,W_2(p)\,W_5(p)\,W_7(p)}{p\,[1+W(p)]}\,\frac{M_z^{\delta}}{J_z}\,\Delta\delta. \end{aligned}\right\} \qquad (3.3)$$

Formulas (3.3) determine transfer functions relating the basic kinematic parameters to the elevator deflection angle. These transfer functions can be analyzed easily by control theory methods, in particular with the help of frequency methods; in addition, in calculation of the frequency characteristics for $[1 + W(p)]$ we can use the well-known diagram [3] which allows us to calculate the frequency characteristics of the closed system when the frequency characteristics of the open system are known.

However, in these calculations, the main advantages of the frequency methods, simplicity and clarity, are lost. Therefore, we shall henceforth simplify all transfer functions, the simplification being based on the fact (see Section II) that the natural frequencies of the individual links are widely separated on the frequency scale. Let us consider first the case of horizontal flight.

Let us compare the roots of the characteristic equation

$$\frac{1}{W_1(p)\,W_2(p)} - W_3(p)\,W_4(p) = 0$$

for the case of a plane with an immobile rudder, with the roots of the equations describing links I and II. The calculations will be carried out for both altitudes, using the previously assumed aerodynamic data.

As a result we obtain two algebraic equations of fourth degree:

at H = 5,000 meters

$$p^4 + 2{,}914p^3 + 8{,}331p^2 + 0{,}1188p + 0{,}0294 = 0,$$

at H = 12,000 meters

$$p^4 + 1{,}299p^3 + 3{,}278p^2 + 0{,}0455p + 0{,}0201 = 0.$$

Table II presents the results of calculation of the roots of the complete characteristic equation, and the roots of polynomials which describe links I and II $\left(\frac{1}{W_1(p)}=0 \text{ and } \frac{1}{W_2(p)}=0\right)$.

TABLE II

Characteristic equations	Roots	H = 5000 m	H = 12000 m
Complete	Large	$-1.45\pm 2.47i$	$-0.644\pm 1.687i$
	Small	$-0.00655\pm 0.0592i$	$-0.0056\pm 0.0783i$
$\frac{1}{W_1(p)}=0$	Link I	$-1.455\pm 2.49i$	$-0.642\pm 1.69i$
$\frac{1}{W_2(p)}=0$	Link II	$-0.007\pm 0.0651i$	$-0.0067\pm 0.082i$

It is known [2] that a pair of large roots characterizes a short-period motion. Calculations of the roots of the complete equation and roots of the quadratic expression $1/W_1(p) = 0$, which characterizes link I, show that the roots in both cases are sufficiently close to one another. This means that the transfer function $W_1(p)$ quite accurately describes the short-period airplane motion.

More than that, in our example the pair of small roots is close in magnitude to two roots of the characteristic equation which describes link II. It should be noted that the sufficiently close proximity of the small roots, which we have obtained in this case, sometimes can not be obtained.

If we are interested only in the short period motion, then in Equation (3.3) we can assume, by way of an approximation,

$$\frac{1}{1+W(p)}=\frac{1}{1-W_1(p)W_2(p)W_3(p)W_4(p)}\approx 1. \tag{3.4}$$

In order to show this, let us rewrite the left side of Equation (3.4) in the following manner;

$$\frac{\frac{1}{W_1(p)}}{\frac{1}{W_1(p)}-W_2(p)W_3(p)W_4(p)}\approx\frac{\frac{1}{W_1(p)}}{\frac{1}{W_1(p)}-\frac{1}{p^2}W_3(p)W_4(p)}. \tag{3.5}$$

Here it is assumed that $W_2(p)\approx\frac{1}{p^2}$ because of following considerations. As it was noted before (see section 2), the natural frequency of link I is several tens of times larger than the natural frequency of link II. An increase in flight velocity leads to an increase of frequency in link I, that is, to an increase in the difference between the frequencies of links I and II. Therefore, link I in the region of frequency ω_1 can be approximately described by the transfer function

$$W_2(p)=\frac{1}{p^2}.$$

It is easy to convince oneself that in most cases the second term in the denominator of (3.5) can be neglected because of the smallness of the product of the amplification factors of links III and IV (see Figs. 3 and 4). Therefore, in this case the characteristic equation can be written approximately in the form $\frac{1}{W_1(p)}=0$. This fact is well known in the literature [1] and is confirmed by the above example.

For the case of short-period motion, the equation of motion (3.2), can be presented in a simplified from as follows:

$$\frac{\Delta\alpha}{W_1(p)} - \frac{M_z^{\delta}}{J_z}\Delta\delta = 0. \tag{3.6}$$

The transfer functions (3.3) for short-period motion can be rewritten, considering the simplified expression (3.4).

As a result we will obtain the following relations:

$$\left.\begin{aligned}
\Delta\alpha &= W_1(p)\frac{M_z^{\delta}}{J_z}\Delta\delta,\\
\Delta V &= \frac{W_1(p)}{p^2}W_3(p)\frac{M_z^{\delta}}{J_z}\Delta\delta,\\
\Delta\theta &= \frac{W_1(p)}{p^2}W_5(p)\frac{M_z^{\delta}}{J_z}\Delta\delta,\\
\Delta H &= \frac{1}{p^3}W_1(p)W_5(p)W_7(p)\frac{M_z^{\delta}}{J_z}\Delta\delta.
\end{aligned}\right| \tag{3.7}$$

Harmonic oscillation of the rudder induces oscillations of the angle $\Delta\alpha$, which will lag in phase the oscillations $\Delta\delta$. The angle of phase shift will increase from 0 to $-180°$ with increase in oscillation frequency from zero to ∞. The amplitude of the forced oscillations of angle $\Delta\alpha$ will increase from the static deflection of $\Delta\alpha$ at $\omega = 0$ to a maximum value at $\omega \approx \omega_1$, and at $\omega > \omega_1$ it will begin to decrease again. These changes are shown in Figs. 2 and 3 [frequency characteristics of the transfer function $W_1(p)$].

Let us consider all the other expression in (3.7). Here we shall replace the multiplier $W_1(p)\frac{M_z^{\delta}}{J_z}\Delta\delta$ by $\Delta\alpha$, that is, we shall consider the changes ΔV, $\Delta\vartheta$*, $\Delta\theta$, and ΔH, assuming that the airplane goes through harmonic oscillations of the angle $\Delta\alpha$.

From the system (3.7) it is easy to see that $\Delta V = W_2(p)W_3(p)\Delta\alpha$. With the aerodynamic data which we have assumed in our calculations, the dependence of ΔV on $\Delta\alpha$ will take the following form (H = 5,000 meters):

$$\Delta V = \frac{8{,}42p + 12{,}67}{p^2}\Delta\alpha. \tag{3.8}$$

As was shown, the natural frequency of airplane vibrations in short-period motion for various examples is within the limits $\omega_1 = 1.8$ to $2.91\ \text{sec}^{-1}$.

If the plane oscillates harmonically through $\Delta\alpha$ with the frequency ω_1, then the change of velocity ΔV will be determined by the transfer function (3.8). The expression in the numerator in this case produces an advance of 62.5°. Thus harmonic oscillations of the angle $\Delta\alpha$ induce velocity vibrations with a phase lag of $62.5° - 180° = -117.5°$. At the amplitude $\Delta\alpha = 0.1$ radians $\approx 5.73°$, the amplitude of velocity change will be small: $\Delta V = 0.3$ m/sec, that is, 0.135% of the initial velocity V = 222 meters per second.

Now let us consider a simplified expression of the transfer function

$$\frac{\Delta\theta}{\Delta\alpha} = W_2(p)W_5(p) \approx \frac{W_5(p)}{p^2}.$$

With the aerodynamic data which we have assumed in our calculation, this transfer function will take form:

* Translator's Note: The equation for $\Delta\vartheta$ was omitted from (3.7). The reader can supply it by reference to (3.3).

$$\frac{\Delta\theta}{\Delta\alpha}=W_2(p)\,W_5(p)\approx\frac{W_5(p)}{p^2}=\frac{1,29p+3\cdot2}{p^2}.$$

Oscillations of the angle of attack with the amplitude $\Delta\alpha$ = 0.1 radians induce oscillations of the angle $\Delta\theta$, which lag in phase by 22° - 180° = -158°, with the amplitude $\Delta\theta$ = 0.0586 radians. Thus, the airplane's oscillation through an angle of attack $\Delta\alpha$ with frequency ω_1 induces oscillations of the airplane's center of gravity, which creates a deviation of the velocity vector by an angle whose amplitude reaches 60% of the amplitude of the angle of attack.

From Relation (3.7)* for $\Delta\vartheta$ it is seen that

$$\Delta\vartheta=\frac{1}{p^2}\,W_6(p)\,\Delta\alpha=\frac{1}{p^2}\,[W_5(p)+p^2]\,\Delta\alpha=\Delta\alpha+\Delta\theta.$$

Thus, the angle $\Delta\vartheta$, in the case when the airplane vibrates harmonically is a sum of two harmonic vibrations; the vibrations of angle $\Delta\alpha$ and angle $\Delta\theta$ (shifted in phase and distorted in the amplitude from the oscillation of $\Delta\alpha$).

Finally, let us look at the altitude oscillation ΔH, when the airplane is undergoing harmonic oscillations of the angle $\Delta\alpha$. From Formula (3.7) for ΔH it is seen that

$$\frac{\Delta H}{\Delta\alpha}=\frac{V}{p}\,W_2(p)\,W_5(p)\,W_7(p)=\frac{VW_5(p)\,W_7(p)}{p^3}.$$

When θ = 0 link VII becomes an amplifying link with the amplification factor k_7 = 1. Therefore, ΔH in this case will be a simple integral of the function $V\cdot\Delta\theta$. This means that the vibrations ΔH will be shifted in relation to $\Delta\alpha$ by the additional angle -90°, and will have the amplitude ΔH = 3.1 meters at an amplitude of $\Delta\alpha$ = 0.1 radian.

Thus, at large frequencies (rapid oscillations) the motion is reduced only to the change of $\Delta\alpha$, $\Delta\vartheta$ and $\Delta\theta$ with other parameters being constant.

Let us now consider the slow airplane motion, which is characterized by the pair of small roots of the characteristic equation.

In this case link I can be approximately considered amplifying with the amplification factor.

$$k_1=W_1(0)=-\frac{J_z}{\overline{M}_z^{\alpha}}=-\frac{\overline{r}_z^2\left(\frac{2m}{\rho SV}\right)^2}{m_z^{\alpha}\frac{2m}{\rho Sb}+m_z^{\bar{\omega}_z}(c_y^{\alpha}-c_x^{\alpha}\operatorname{tg}\theta)}. \qquad (3.10)$$

This property of link I allows us to simplify significantly Formulas (3.3) for the case of slow motion. Let us first examine the characteristic equation

$$\frac{1}{W_1(p)\,W_2(p)}-W_3(p)W_4(p)=0$$

considering the simplification (3.10). Let us consider the case of horizontal flight. For horizontal flight θ = 0 and the characteristic equation will assume the form

$$A_0p^2+A_1p+A_2=0, \qquad (3.11)$$

where

* See Translator's note, preceding page.

$$\left.\begin{aligned} A_0 &= -\frac{X^\alpha}{m}\frac{Y^V}{mV} - \frac{M_z^\alpha}{J_z} - \frac{M_z^{\omega_z}}{J_z}\frac{Y^\alpha}{mV}, \\ A_1 &= -g\frac{Y^\alpha}{mV}\frac{Y^V}{mV} + \frac{X^\alpha}{m}\frac{M_z^V}{J_z} - \frac{X^V}{m}\frac{M_z^\alpha}{J_z} + \frac{M_z^{\omega_z}}{J_z}\left[\frac{X^\alpha}{m}\frac{Y^V}{m} - \frac{X^V}{m}\frac{Y^\alpha}{m}\right]\frac{1}{V}, \\ A_2 &= \left[\frac{Y^\alpha}{m}\frac{M_z^V}{J_z} - \frac{Y^V}{m}\frac{M_z^\alpha}{J_z}\right]\frac{g}{V}. \end{aligned}\right\} \tag{3.12}$$

Let us insert into Equation (3.11) the above aerodynamic coefficients. We will obtain the following equations:

1) For H = 5,000 meters

$$p^2+0{,}0128p+0{,}00354=0$$

with the roots $p_{1,2} = -0.0064 \pm 0.0597i$.

2) For H = 12,000 meters

$$p^2+0{,}0117p+0{,}00636=0$$

with the roots $p_{1,2} = -0.0058 \pm 0.0793i$.

These roots are very close to the small roots of the characteristic equation, which have been calculated previously for the complete system of equations, and do not differ very much from the poles of the transfer function of link II.

The equations of longitudinal motion, (3.2), when combined with the simplification (3.10) for the case of slow motion can be expressed in the following form:

$$\left.\begin{aligned} \Delta\alpha &= -\frac{J_z}{M_z^\alpha}\left[W_4(p)\,\Delta V + \frac{M_z^\delta}{J_z}\Delta\delta\right], \\ \Delta V &= W_2(p)\,W_3(p)\,\Delta\alpha. \end{aligned}\right\} \tag{3.13}$$

This system of equations is of the second order.

Let us rewrite Equations (3.3) considering the simplification (3.10). Prior to that let us divide the denominator and numerator of Formulas (3.3) by $W_2(p)$. This transformation, with the consideration of simplification (3.10) changes the denominators of Formulas (3.3) into quadratic expressions. Thus, the expression in the denominator can be considered a transfer function of a certain new link, which is close to link II in its characteristics. We shall call this link a long-period link, and it will be of great importance in the following. As a result we will obtain the following formulas:

$$\Delta\alpha = -\frac{\dfrac{1}{W_2(p)}}{\dfrac{1}{W_2(p)} - \dfrac{J_z}{M_z^\alpha}W_3(p)\,W_4(p)}\,\frac{M_z^\delta}{M_z^\alpha}\Delta\delta,$$

$$\Delta V = -\frac{W_3(p)}{\dfrac{1}{W_2(p)} - \dfrac{J_z}{M_z^\alpha}W_3(p)\,W_4(p)}\,\frac{M_z^\delta}{M_z^\alpha}\Delta\delta,$$

$$\Delta\vartheta = -\frac{W_6(p)}{\dfrac{1}{W_2(p)} - \dfrac{J_z}{M_z^\alpha}W_3(p)\,W_4(p)}\,\frac{M_z^\delta}{M_z^\alpha}\Delta\delta$$

$$\Delta\theta = -\frac{W_5(p)}{\frac{1}{W_2(p)} - \frac{J_z}{M_z^\alpha} W_3(p)\,W_4(p)} \frac{M_z^\delta}{M_z^\alpha} \Delta\delta,$$

$$\Delta H = -\frac{1}{p} \frac{W_5(p)\,W_7(p)}{\frac{1}{W_2(p)} - \frac{J_z}{M_z^\alpha} W_3(p)\,W_4(p)} \frac{M_z^\delta}{M_z^\alpha} \Delta\delta. \qquad (3.14)$$

By introducing into these formulas the aerodynamic data which have been assumed in previous calculations, we will obtain the following relation between $\Delta\alpha$ and $\Delta\delta$:

$$\Delta\alpha = -\frac{p^2+0,014p+0,00428}{p^2+0,014p+0,00428-\frac{1}{6,86}(8,42p+12,7)(0,000437p+0,000485)} \frac{5,35}{6,86}\Delta\delta =$$

$$= -\frac{p^2+0,014p+0,00428}{(1-0,000538)\,p^2+(0,014-0,0014)\,p+0,00428-0,0009} \frac{5,35}{6,86}\Delta\delta.$$

It is easily seen that the influence of links III and IV in this case is insignificant. If we neglect this influence, then we will have a direct proportionality between the displacement of the elevator and the deviation of the airplane by the angle

$$\Delta\alpha \approx -\frac{M_z^\delta}{M_z^\alpha}\Delta\delta.$$

At frequencies which are close to the natural frequency of the airplane during slow motion, we will have a significant change in phase and amplitude of the induced oscillations of the attack angle, in relation to the oscillations of the elevator. As is shown in Fig. 7, the amplitude of $\Delta\alpha$ is subject to two sharp changes with the change of frequency of $\Delta\delta$: at frequency ω_2, equal to the reference frequency of link II, the amplitude of $\Delta\alpha$ reaches its maximum; at ω_2', equal to the reference frequency of the link which we had previously called the long period one, the amplitude of $\Delta\alpha$ reaches its minimum. Further increase in frequency leads to proportionality between the deviations $\Delta\alpha$ and $\Delta\delta$. In exactly the same way the proportionality is preserved at low frequencies. The sharp change in amplitude from maximum to minimum is accompanied, as can be seen from the frequency characteristic in Fig. 7, by a phase lag of the oscillations of $\Delta\alpha$ with respect to the oscillations of $\Delta\delta$, and at the frequency corresponding to the point of inflection of the amplitude-frequency characteristic, this phase shift attains a magnitude of 45°.

Now let us consider the change of airplane velocity during a slow displacement of the elevator. In our case this change will be determined by the following transfer function:

$$\frac{\Delta V}{V} = \frac{0,038p+0,057}{p^2+0,0126p+0,00338} \frac{5,36}{6,86}\Delta\delta.$$

In the numerator of this function there is an expression in which at $p = i\omega_2 = 0,06i$ we can disregard the influence of the imaginary part; in other words at low frequencies link III can be considered as a purely amplifying link with an amplification factor of 0.057.

In the denominator, there is an expression which describes a long-period link. As we have already shown, the natural frequency of this link almost equals the frequency of the slow oscillation of the airplane. Therefore, and also by considering everything that has been said about the numerator of the transfer function, we can say that the oscillations of the velocity increment ΔV lag the oscillations of the elevator: in particular, at the natural frequency of the oscillations the lag angle equals 90°. The amplitude of these vibrations in comparison to the static deviation ΔV increases to a certain maximum, and then decreases; its maximum ratio to the static value is equal to the ratio between the natural frequency and twice the time it takes the vibtation to decrease to 1/e times its original value (coefficient of p).

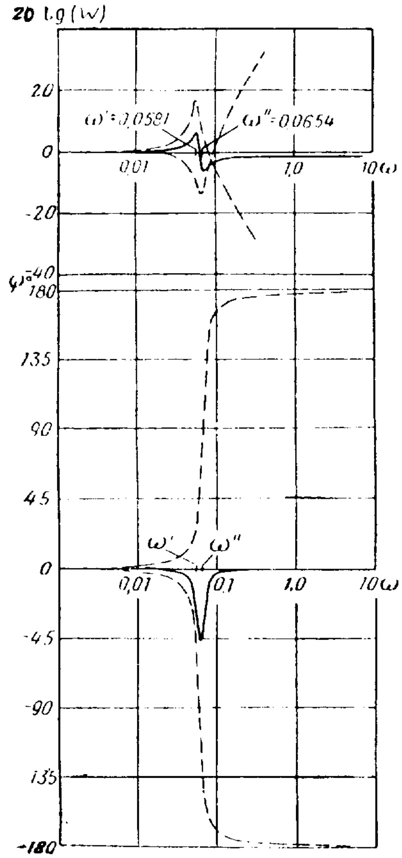

Fig. 7. Logarithmic frequency characteristics of the approximate transfer function of the elevator control system, in the case of slow longitudinal motion. Altitude 12 kilometers, velocity 800 kilometers per hour.

In our case this value is equal to

$$\frac{\sqrt{0{,}00338}}{0{,}0126} = 4{,}6.$$

As has been said before, the long-period link is close in its characteristics to link II, because the influence of links III and IV is small. Therefore, the damping coefficient is approximately equal to $\frac{1}{2}\frac{X^V}{m}$. From this it is easily seen that a decrease of X^V (thrust increase with speed increase in turbojet engines at supersonic speeds) will lead to an increase in the ratio of the velocity amplitude to the frequency of the slow natural oscillations of the airplane.

Similar reasoning can be applied to the change in $\Delta\theta$, because the transfer function of link V in our case can be regarded as a function of a purely amplifying link. However, in general, with large angles of attack, as has been discussed before, the reference frequency of link V, which as a rule decreases with decrease of speed, can be greatly changed. Link V can even become unstable. A decrease of the reference frequency of link V results in a proportional decrease of the amplitude of induced oscillation $\Delta\theta$.

Let us consider the change in altitude during slow deviation of the elevator $\Delta\delta$: during horizontal flight $\theta = 0$ and, therefore, $W_7(p) = 1$, and ΔH will be a simple integral of the change $V\Delta\theta$; thus the change of altitude will be completely determined by the character of the change of the trajectory angle $\Delta\theta$.

In conclusion, let us note that in the slow motion the change of the parameters $\Delta\theta$, ΔV and ΔH depends largely upon the qualities of link II, for which the slow-motion frequency is almost the resonance frequency, and therefore in this case small changes of the frequency ω of slow motion leads to a large shift in phase of the induced oscillations. Here the character of the induced oscillations depends largely on the derivative with respect to flight velocity of the tangential forces are applied to the airplane:

$$\frac{X^V}{m} - \frac{g}{V}\sin\theta.$$

Let us now turn to the case when $\theta \neq 0$. In this case the transfer function $W_4(p)$ is a polynomial of second order, and not of the first as in the case of $\theta = 0$, which seems to cause some complications.

At large frequencies which are in the neighborhood of the reference frequency ω_1 of link I, we can consider, as before, that link II is doubly-integrating with transfer function $W_2(p) = \frac{1}{p^2}$, link III is purely differentiating with transfer function $W_3(p) = -\frac{X^\alpha}{m}p$, and link IV' is purely differentiating with transfer function $\underline{p}$. Therefore we can write

$$\frac{1+W(p)}{W_1(p)} = \frac{1}{W_1(p)} - W_2(p)\,W_3(p)\,W_4'(p)\,W_4''(p) \approx \frac{1}{W_1(p)} + \frac{X^\alpha}{m}\,W_4''(p).$$

Expanding this expression, we find (in the case when $M_Z^V = 0$)

$$\frac{1+W(p)}{W_1(p)}=p^2+\left(\frac{\overline{Y}^\alpha}{mV\cos\theta}-\frac{M_z^{\omega}+M_z^{\dot\alpha}}{J_z}\right)p-\frac{\overline{M}_z^\alpha}{J_z}+\frac{X^\alpha}{m}\left(p\,\frac{\operatorname{tg}\theta}{m}-\frac{\overline{Y}^V}{mV\cos\theta}\right)=$$

$$=p^2+\left(\frac{Y^\alpha}{mV}-\frac{M_z^{\omega_z}+M_z^{\dot\alpha}}{J_z}\right)p-\frac{M_z^\alpha}{J_z}-\frac{Y^\alpha}{mV}\frac{M_z^{\omega_z}}{J_z}-\frac{X^\alpha}{mV}\left(\frac{M_z^{\omega_z}}{J_z}+\frac{X^V}{m}\right)\operatorname{tg}\theta. \quad (3.15)$$

From this expression we can see that for short-period motion, even at a large angle θ, the damping coefficient does not change, but the frequency changes somewhat with the change of angle θ. However, since the multiplier of $\tan\theta$ is very small, then even for large θ, up to 80°, we can consider that the short period motion is completely described by link I alone.

In the case of slow (long-period) motion we can again consider

$$W_1(p)=k_1=-\frac{J_z}{M_z^\alpha},\qquad W_4'(p)=-\frac{M^{\omega_z}}{J_z}.$$

With these conditions we have

$$\frac{1+W(p)}{W_2(p)}=\frac{1}{W_2(p)}-W_1(p)\,W_3(p)\,W_4'(p)\,W_4''(p)=\frac{1}{W_2(p)}-\frac{M_z^{\omega_z}}{\overline{M}_z^\alpha}M_3(p)\,W_4''(p)=$$

$$=p^2+\left(\frac{X^V}{m}-\frac{g}{V}\sin\theta\right)p+\frac{\overline{Y}^V g}{mV}-\frac{M_z^{\omega_z}}{\overline{M}_z^\alpha}\left[\frac{X^\alpha}{m}p+\frac{Y^\alpha p}{mV}\right]\left[p\,\frac{\operatorname{tg}\theta}{V}-\frac{Y^V}{mV\cos\theta}\right]. \quad (3.16)$$

Therefore, also in the case of long-period motion, we can approximately consider the airplane as a system of second order, as before. But, in contrast to the short period motion, the presence of an angle θ very strongly influences the transfer function during control with the elevator.

In order to illustrate and confirm the degree of approximation, the roots of the complete characteristic equation for the case of $\theta = 60°$ (H = 12 kilometers) were calculated, as well as the roots of the polynomials (3.15) and (3.16). The results are shown in Table III.

TABLE III

Kind of motion	Exact values of roots	By Formulas (3.15) and (3.16)	Roots of polynomials $\frac{1}{W_1(p)}$ and $\frac{1}{W_2(p)}$
Long-period	$0.0107 \pm 0.495i$	$0.0114 \pm 0.0494i$	$0.0124 \pm 0.0515i$
Short-period	$-0.513 \pm 1.74i$	-0.478 ± 1.752	$-0.478 \pm 1.732i$

As can be seen from this table, the exact and the approximated values of the roots are very close. Also the approximated values of the roots of the characteristic equation can be obtained as roots of the polynomials $\frac{1}{W_1(p)}$ and $\frac{1}{W_2(p)}$, but with less accuracy.

All the previous considerations are true when $M_Z^V = 0$. When $M_Z^V \neq 0$, then the general considerations remain in force, but in this case $W_4(p)$ has to be broken down into its factor

$$W_4(p)=\left(p\,\frac{\operatorname{tg}\theta}{V}-p_1\right)(p-p_2),$$

in which the relationship of frequencies will usually be preserved. Therefore, during short-period vibrations

$$W_4(p) = p\left(p\frac{\operatorname{tg}\theta}{V} - p_1\right);$$

during long-period vibrations

$$W_4(p) = -p_1(p - p_2).$$

CHAPTER II

LATERAL MOTION

Section 4. Block Diagram of Perturbed Lateral Motion

Let us write the equations of lateral motion in operator form* (see Appendix I) [2] **

$$\left.\begin{aligned}
&\left(p-\frac{Z^{\beta}}{mV}\right)\beta-\left(p\sin\alpha+\frac{g}{V}\cos\theta\right)\gamma-\omega_{y}\frac{\cos\theta}{\cos\vartheta}=\frac{Z^{\delta_H}}{mV}\delta_{H},\\
&-\frac{M_x^{\beta}}{J_x}\beta+\left(p^2-\frac{M_x^{\omega_x}}{J_x}p\right)\gamma+\left[\operatorname{tg}\vartheta p-\left(\frac{M_x^{\omega_y}}{J_x}+\frac{M_x^{\omega_x}}{J_x}\operatorname{tg}\vartheta\right)\right]\omega_y=\frac{M_x^{\delta_{\text{Э}}}}{J_x}\delta_{\text{Э}}+\frac{M_x^{\delta_H}}{J_x}\delta_H,\\
&-\frac{M_y^{\beta}}{J_y}\beta-\frac{M_y^{\omega_x}}{J_y}p\gamma+\left[p-\left(\frac{M_y^{\omega_y}}{J_y}+\frac{M_y^{\omega_x}}{J_y}\operatorname{tg}\vartheta\right)\right]\omega_y=\frac{M_y^{\delta_{\text{Э}}}}{J_y}\delta_{\text{Э}}+\frac{M_y^{\delta_H}}{J_y}\delta_H,\\
&\qquad p\psi=\frac{\omega_y}{\cos\vartheta},\\
&\qquad p\gamma=\omega_x-\sin\vartheta p\psi=\omega_x-\omega_y\operatorname{tg}\vartheta.
\end{aligned}\right\}\quad(4.1)$$

Let us eliminate the angle of sideslip from the system of equations (4.1). In order to do that we have to add the first equation, after it has been multiplied by $\frac{M_y^{\beta}}{J_y}$, to the third equation, which has been multiplied by $\left(p-\frac{Z^{\beta}}{mV}\right)$, the second equation, multiplied by $-\frac{M_y^{\beta}}{J_y}$, we must add to the third equation multiplied by $\frac{M_x^{\beta}}{J_x}$; we will obtain

$$\left.\begin{aligned}
&\left[p^2-\left(\frac{Z^{\beta}}{mV}+\frac{M_y^{\omega_y}}{J_y}+\frac{M_y^{\omega_x}}{J_y}\operatorname{tg}\vartheta\right)p+\frac{Z^{\beta}}{mV}\left(\frac{M_y^{\omega_y}}{J_y}+\frac{M_y^{\omega_x}}{J_y}\operatorname{tg}\vartheta\right)-\frac{M_y^{\beta}}{J_y}\frac{\cos\theta}{\cos\vartheta}\right]\omega_y=\\
&=\left[\frac{M_y^{\omega_x}}{J_y}p^2+\left(-\frac{M_y^{\omega_x}}{J_y}\frac{Z^{\beta}}{mV}+\frac{M_y^{\beta}}{J_y}\sin\alpha\right)p+\frac{g}{V}\frac{M_y^{\beta}}{J_y}\cos\theta\right]\gamma-\frac{M_y^{\delta_{\text{Э}}}}{J_y}\left(p-\frac{Z^{\beta}}{mV}\right)\delta_{\text{Э}}+\\
&\qquad+\left[\frac{M_y^{\delta_H}}{J_y}p-\frac{M_y^{\delta_H}}{J_y}\frac{Z^{\beta}}{mV}+\frac{Z^{\delta_H}}{mV}\frac{M_y^{\beta}}{J_y}\right]\delta_H,\\
&\left[-\frac{M_y^{\beta}}{J_y}p+\frac{M_y^{\beta}}{J_y}\frac{M_x^{\omega_x}}{J_x}-\frac{M_x^{\beta}}{J_x}\frac{M_y^{\omega_x}}{J_y}\right]p\gamma=\left[\left(-\frac{M_x^{\beta}}{J_x}+\frac{M_y^{\beta}}{J_y}\operatorname{tg}\vartheta\right)p-a_{4,0}\right]\omega_y+\\
&\qquad+\left(-\frac{M_x^{\delta_{\text{Э}}}}{J_x}\frac{M_y^{\beta}}{J_y}+\frac{M_y^{\delta_{\text{Э}}}}{J_y}\frac{M_x^{\beta}}{J_x}\right)\delta_{\text{Э}}+\left(-\frac{M_x^{\delta_H}}{J_x}\frac{M_y^{\beta}}{J_y}+\frac{M_y^{\delta_B}}{J_y}\frac{M_x^{\beta}}{J_x}\right)\delta_H,
\end{aligned}\right\}\quad(4.2)$$

---

* It is assumed that the coordinate axes here are the principal axes of inertia, and all derivatives are referred to these principal axes (see Appendix II).

** $\delta_{\text{Э}}$ and δ_H are standard Russian symbols for deflection of the ailerons and rudder, respectively. Translator's Note.

where

$$a_{4,0} = -\frac{M_x^\beta}{J_x}\left(\frac{M_y^{\omega_y}}{J_y} + \frac{M_y^{\omega_x}}{J_y}\operatorname{tg}\vartheta\right) + \frac{M_y^\beta}{J_y}\left(\frac{M_x^{\omega_y}}{J_x} + \frac{M_x^{\omega_x}}{J_x}\operatorname{tg}\vartheta\right). \tag{4.3}$$

The second of these equations possesses a simple physical meaning. Let us write the moment equations about an axis which is inclined to the longitudinal principal axis by an angle φ. For this, it is sufficient to multiply the second equation by $J_x \cos\varphi$, the third by $J_y \sin\varphi$, and add them. Here the terms containing β will assume the form

$$(M_x^\beta \cos\varphi + M_y^\beta \sin\varphi)\,\beta = \overline{M}^\beta \beta,$$

where $\overline{M}$ is the moment about the new axis. If we choose the angle according to the equation

$$\operatorname{tg}\varphi = -\frac{M_x^\beta}{M_y^\beta},$$

we will obtain exactly the second equation of (4.2), in which in this case, $\overline{M}^\beta = \dfrac{\partial\overline{M}}{\partial\beta} = 0$. In other words, the second equation of (4.2) represents nothing else but the moment equation about an axis such that the moment about this axis does not depend on the angle of sideslip. This axis can be called "the neutral axis of sideslip".

The system of equations (4.2) can be presented in the form of the block diagram, which is shown in Fig. 8; the transfer functions of the individual links have the form

$$\left.\begin{aligned}
W_1(p) &= \frac{1}{p^2 - \left(\dfrac{Z^\beta}{mV} + \dfrac{M_y^{\omega_y}}{J_y} + \dfrac{M_y^{\omega_x}}{J_y}\operatorname{tg}\vartheta\right)p + \dfrac{Z^\beta}{mV}\left(\dfrac{M_y^{\omega_y}}{J_y} + \dfrac{M_y^{\omega_x}}{J_y}\operatorname{tg}\vartheta\right) - \dfrac{M_v^\beta}{J_y}\dfrac{\cos\theta}{\cos\vartheta}},\\
W_2(p) &= \left(-\frac{M_x^\beta}{J_x} + \frac{M_y^\beta}{J_y}\operatorname{tg}\vartheta\right)p - a_{4,0},\\
W_3(p) &= \frac{1}{-\dfrac{M_v^\beta}{J_y}p + \dfrac{M_y^\beta}{J_y}\dfrac{M_x^{\omega_x}}{J_x} - \dfrac{M_x^\beta}{J_x}\dfrac{M_y^{\omega_x}}{J_y}},\\
W_4(p) &= \frac{1}{p},\\
W_5(p) &= \frac{M_y^{\omega_x}}{J_y}p^2 + \left(-\frac{M_y^{\omega_x}}{J_y}\frac{Z^\beta}{mV} + \frac{M_y^\beta}{J_y}\sin\alpha\right)p + \frac{g}{V}\frac{M_v^\beta}{J_y}\cos\vartheta,\\
W_6(p) &= \frac{M_v^{\delta_H}}{J_y}p - \frac{M_v^{\delta_H}}{J_y}\frac{Z^\beta}{mV} + \frac{M_y^\beta}{J_y}\frac{Z^{\delta_H}}{mV},\\
W_7(p) &= \frac{M_v^{\delta_э}}{J_y}\left(p - \frac{Z^\beta}{mV}\right).
\end{aligned}\right\} \tag{4.4}$$

The system of links I – V forms a closed stabilization loop. The other links will be called the steering links (cf. Section I).

Frequently it is necessary to introduce the signal β. In determining β from the third equation of the system (4.1), after simple but extensive calculations we will find

$$\beta = W_1(p)\left\{W_8(p)\gamma + \frac{M_y^{\delta_э}}{J_y}\frac{\cos\theta}{\cos\vartheta}\delta_э + \left[\frac{Z^{\delta_H}}{mV}p + \frac{M_y^{\delta_H}}{J_y}\frac{\cos\theta}{\cos\vartheta} - \frac{Z^{\delta_H}}{mV}\left(\frac{M_y^{\omega_y}}{J_y} + \frac{M_y^{\omega_x}}{J_y}\operatorname{tg}\vartheta\right)\right]\delta_H\right\}, \tag{4.5}$$

where

$$W_8(p) = \sin\alpha\, p^2 + \left(-\frac{M_y^{\omega_y}}{J_y}\sin\alpha + \frac{M_y^{\omega_x}}{J_y}\cos\alpha + \frac{g}{V}\cos\vartheta\right)p - \frac{g}{V}\cos\vartheta\left(\frac{M_y^{\omega_y}}{J_y} + \frac{M_y^{\omega_x}}{J_y}\operatorname{tg}\vartheta\right). \quad (4.6)$$

Equation (4.5) can also be represented in the block diagram Fig. 8. For this it is necessary to send the signal γ through link VIII with the transfer function $W_8(p)$ and add to it the signals $\delta_Э$ and δ_H, sent through the additional links according to Formula (4.5), and the whole should be sent through a link with transfer function $W_1(p)$.

Link VIII with the transfer function $W_8(p)$ is a differentiating link of second order. At $\alpha = 0$ one of the the roots of the polynomial $W_8(p)$ becomes infinite, and the link becomes a differentiating link of the first order with the transfer function

$$W_8(p) = \left(\frac{M_y^{\omega_x}}{J_y} + \frac{g}{V}\cos\vartheta\right)p - \frac{g}{V}\cos\vartheta\left(\frac{M_y^{\omega_y}}{J_y} + \frac{M_y^{\omega_x}}{J_y}\operatorname{tg}\vartheta\right). \quad (4.6')$$

In other words, in this particular case link VIII can be expressed in the form of a sequence of two links in tandem: one amplifying link with the amplification factor

$$k_8' = \frac{M_y^{\omega_x}}{J_y} + \frac{g}{V}\cos\vartheta$$

and a second link with a transfer function

$$W_8'(p) = p - \frac{g}{V}\cos\vartheta\,\frac{\dfrac{M_y^{\omega_y}}{J_y} + \dfrac{M_y^{\omega_x}}{J_y}\operatorname{tg}\vartheta}{\dfrac{M_y^{\omega_x}}{J_y} + \dfrac{g}{V}\cos\vartheta}.$$

If $\alpha \neq 0$, but the value of α is small, then the frequency of the first link becomes finite but very large, and practically, the relations remain: a sequence of two links, one of which practically is amplifying and the other differentiating of the first order. In other words, for small values of α it is possible, in the expression for $W_8(p)$, to set $\alpha = 0$ without a large error, and to make use of Formula (4.6').

Let us note that in the derivation of Equations (4.2), in eliminating β, we are dividing the equation by $\frac{M_y^\beta}{J_y}$; therefore, the block diagram presented is true only for airplanes which have directional stability. This can also be seen from Formulas (4.4); when $\frac{M_y^\beta}{J_y} = 0$ we have

$$W_1(p) = \frac{1}{\left[p - \dfrac{Z^\beta}{mV}\right]\left[p - \left(\dfrac{M_y^{\omega_y}}{J_y} + \dfrac{M_y^{\omega_x}}{J_y}\operatorname{tg}\vartheta\right)\right]},$$

$$W_2(p) = -\frac{M_x^\beta}{J_x}\left[p - \left(\frac{M_y^{\omega_y}}{J_y} + \frac{M_y^{\omega_x}}{J_y}\operatorname{tg}\vartheta\right)\right],$$

$$W_3(p) = -\frac{1}{\dfrac{M_x^\beta}{J_x}\,\dfrac{M_y^{\omega_x}}{J_y}},$$

$$W_4(p) = \frac{1}{p},$$

$$W_5(p) = \frac{M_y^{\omega_x}}{J_y}\,p\left(p - \frac{Z^\beta}{mV}\right).$$

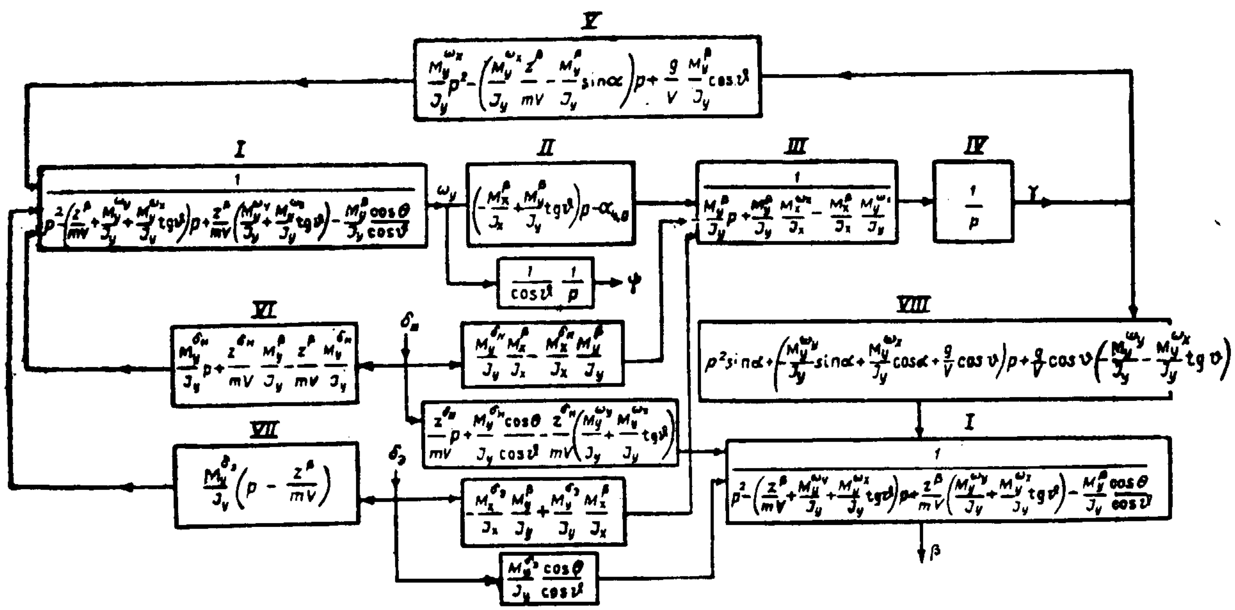

Fig. 8. Block diagram of the system of equations of perturbed lateral motion.

The transfer function W (p) of the open stabilization loop in this case equals

$$W(p) = -W_1(p)\,W_2(p)\,W_3(p)\,W_4(p)\,W_5(p) \equiv -1.$$

Therefore, the block diagram in this case is trivial, and does not give us any significant results.

It is possible to construct another structural scheme for the case $M_y^\beta = 0$; however at this time all aircraft always have directional stability (normally they are even aperiodically unstable), and therefore in this work we are limiting ourselves to the example examined.

Link I is normally an oscillating link; as will be shown in Section 6, this link specifies the presence of a lateral oscillating motion ("the Dutch roll"). We shall call it the oscillating link. Link II is a differentiating link of the first order; the presence of it specifies a spiral motion, and therefore we can call it the spiralling link.

Finally, link III is a link of the first order; the presence of this link, as is clear from the structure of its transfer function, is connected with $M_x^{\omega_x}$, the damping moment of rolling motion; therefore we can call it the rolling member.

As will be shown in Section 6, links VI and VII are in practice either unimportant (at low frequencies) or degenerate into purely differentiating links (at high frequencies); link VII is practically unimportant because of the small magnitude of the coefficient $\dfrac{M_y^{\delta э}}{J_y}$.

In the next section we will give a brief analysis of the characteristics of the links.

Section 5. Brief Analysis of the Characteristics of the Links.

In order to present the characteristics of the individual links comprising the stabilization loop, and also of the control links of the block diagram of lateral motion in contemporary airplanes, we have made calculations of their amplitude-phase characteristics.

TABLE IV

Designation	$H=5$ km	$H=12$ km	Designation	$H=5$ km	$H=12$ km
$q = \dfrac{\rho V^2}{2}\left[\dfrac{kg}{m^2}\right]$	1850	780	$\dfrac{M_y^{\omega_y}}{J_y}\left[\dfrac{1}{sec}\right]$	−0.456	−0.19
$M = \dfrac{V}{a}$	0.69	0.75	$\dfrac{M_x^{\omega_y}}{J_x}\left[\dfrac{1}{sec}\right]$	−1.26	−0.56
$\dfrac{Z^{\delta_H}}{mV}\left[\dfrac{1}{sec}\right]$	−0.0486	−0.0152	$\dfrac{M_x^{\omega_x}}{J_x}\left[\dfrac{1}{sec}\right]$	−3.92	−1.66
$\dfrac{Z^{\beta}}{mV}\left[\dfrac{1}{sec}\right]$	−0.14	−0.059	$\dfrac{M_x^{\delta_H}}{J_x}\left[\dfrac{1}{sec}\right]$	−1.78	−0.75
$\dfrac{M_x^{\beta}}{J_x}\left[\dfrac{1}{sec}\right]$	−14.7	−6.2	$\dfrac{M_x^{\delta э}}{J_y}\left[\dfrac{1}{sec}\right]$	−12.8	−5.7
$\dfrac{M_y^{\beta}}{J_y}\left[\dfrac{1}{sec}\right]$	−5.41	−2.28	$\dfrac{M_y^{\delta_H}}{J_y}\left[\dfrac{1}{sec}\right]$	−1.98	−0.835
$\dfrac{M_y^{\omega_x}}{J_y}\left[\dfrac{1}{sec}\right]$	−0.0442	−0.0198	$\dfrac{M_y^{\delta э}}{J_y}\left[\dfrac{1}{sec}\right]$	0	0

In these calculations we have assumed the following values of the aerodynamic coefficients for the velocity V = 222 meters per second (Table IV).

In the initial regime, for simplicity, it was assumed that $\vartheta = 0$, $\alpha = 0$.

Figure 9 presents, for an altitude of 5 kilometers, the amplitude-frequency and phase-frequency logarithmic characteristics of link I – V, which form the stabilization loop. Analogous characteristics for an altitude of 12 kilometers are presented in Fig. 10. In constructing the characteristics, each transfer function was divided by the amplification factor of the link, i.e., the amplification factor of all links was assumed to be unity. The amplification factors of the links are presented in the same figures.

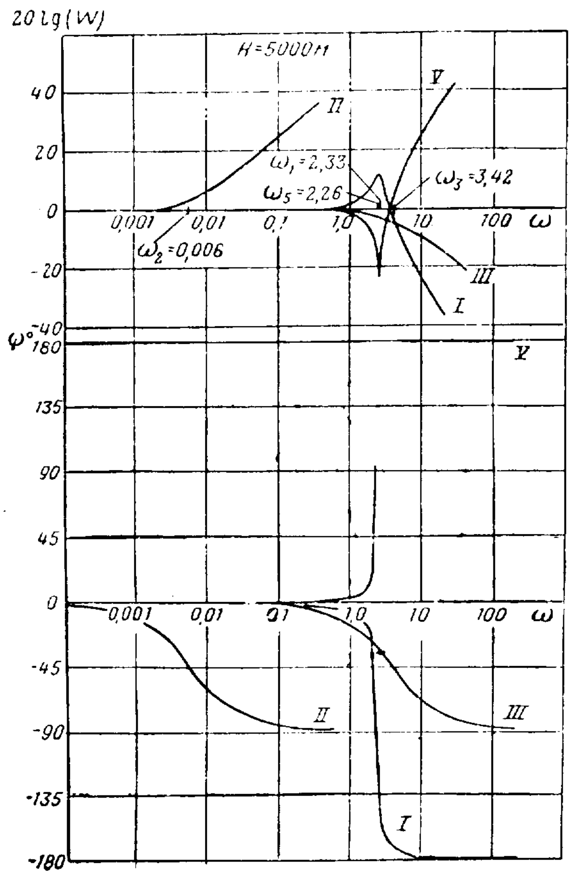

Fig. 9. Logarithmic frequency characteristics of stabilization loop links for lateral motion. Altitude 5,000 m., velocity 800 km/hr. Amplification factors: k_1 = (0.184 sec^2) -14.7 db); k_2 = (0.0881/sec^3) (–22 db); k_3 = (0.054 sec^3) (–25.2 db); k_4 =(1 sec); k_5 = (–0.239/sec^3) (–12.4 db); $k = -k_1k_2k_3k_4k_5$ = 0.000209 (–73.3 db).

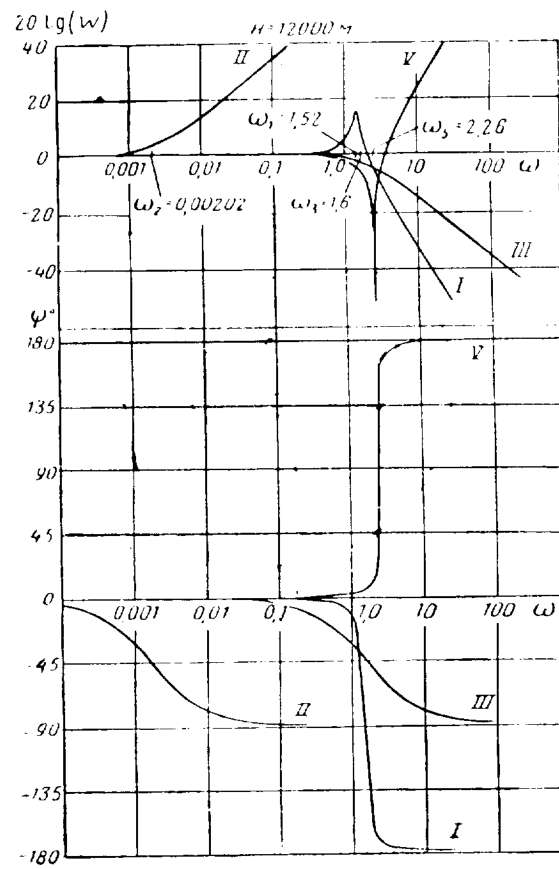

Fig. 10. Logarithmic frequency characteristics of stabilization loop links for lateral motion. Altitude 12,000 m., velocity 800 km/hr. Amplification factors: k_1 = (0.433 sec^2) (–7.29 db); k_2 = 0.0122/sec^3) (–38.2 db); k_3 = (0.274 sec^3) (–11.2 db); k_4 =(1/sec); k_5 = (–0.101/sec^3) (–19.9 db); $k = -k_1k_2k_3k_4k_5$ = (0.0001462) (–76.6 db).

As the calculations have shown, link I with transfer function $W_1(p)$ is a stable oscillating link with reference frequency ω_1:

For H = 5 km, $\omega_1 = 2.33$;

For H = 12 km, $\omega_1 = 1.52$.

The relative damping coefficient of this link equals $\zeta_1 = 0.126$ at an altitude of 5 kilometers, and $\zeta_1 =$ $= 0.081$ at an altitude of 12 kilometers. Thus, this link is weakly damped. In the general theory of dynamic stability [2], this corresponds to the fact that oscillating motion is slowly damped.

Link II with the transfer function $W_2(p)$ is for this example, an unstable differentiating member of first order with reference frequency $\omega_2 = 0.0056$ for altitude 5 km and $\omega_2 = 0.00202$ for altitude 12 km. As is seen from these results, the frequency ω_2 is very small over the entire range of the altitudes considered; and with increase of altitude ω_2 decreases considerably. The argument of the function $W_2(p)$ varies from minus 180° to minus 270°. Thus the airplane under consideration is aperiodically unstable. However, the coefficient $a_{4,0}$ which determines this instability is very small. Therefore, because $a_{4,0}$ is small, the frequencies ω_2 are also small.

The presence of aperiodic instability is typical not only for the airplane under consideration; most contemporary aircraft are aperiodically unstable.

The link of first order with transfer function $W_3(p)$ is stable. Its reference frequencies for the cases H = 5 km and H = 12 km, respectively, are $\omega_3 = 3.42 \frac{1}{\sec}$ and $\omega_3 = 1.6 \frac{1}{\sec}$. As can be seen from these results, the reference frequency changes considerably with increase of flight altitude.

The differentiating link V of the second order with transfer function $W_5(p)$ is a stable link with reference frequency $\omega_5 = 2.26$; the relative damping coefficient of this link is $\zeta_5 = 0.00751$ at an altitude of 5 km, and $\zeta_5 = 0.00326$ at an altitude of 12 km. Thus, this link possesses a very small damping coefficient. Practically, it is a harmonic link.

Figures 11 and 12 show amplitude-frequency and phase-frequency characteristics of the control links. As in the first case, the amplification factor of these links was assumed to be one. Since at the altitude H = = 12 km, $\omega_8 = \infty$, the frequency characteristic of this link is not shown in Fig. 11.

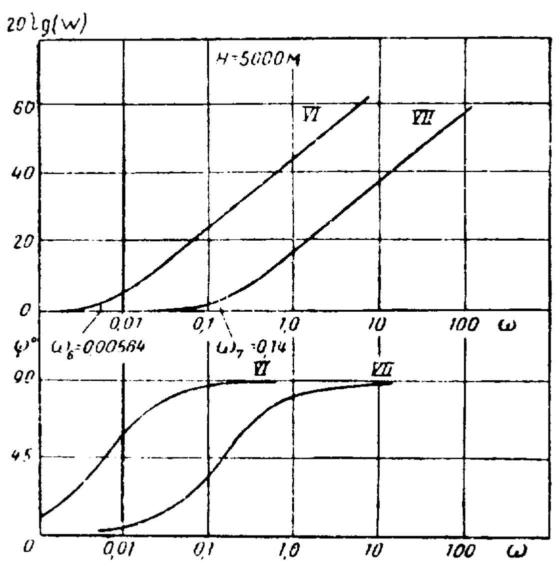

Fig. 11. Logarithmic frequency characteristic of stabilization loop links for lateral motion. Altitude 5,000 m, velocity 800 km/hr. Amplification factors: $k_6 = (-0.0112/\sec^3)$ (-39 db); $k_7 = 0$; $k_8 = (0.0202/\sec^2)$ (-33.9 db).

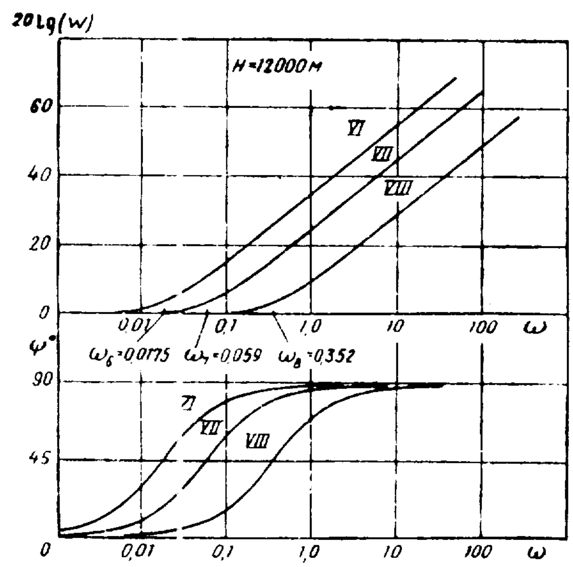

Fig. 12. Logarithmic frequency characteristics of stabilization loop links for lateral motion. Altitude 12,000 m., velocity 800 km/hr. Amplification factors: $k_6 = (0.0147/\sec^3)$ (-36.6 db); $k_7 = 0$; $k_8 = (0.0084/\sec^2)$ (-45 db).

The control links with transfer functions $W_6(p)$, $W_7(p)$, $W_8(p)$ are stable differentiating links of the first order. Their reference frequencies are equal to:

At atitude 5 km	At altitude 12 km
$\omega_6 = 0.00564$	$\omega_6 = 0.0175$
$\omega_7 = 0.14$	$\omega_7 = 0.059$
$\omega_8 = \infty$	$\omega_8 = 0.346$

As it seen from these results, the frequencies of the control loop links (except for link VIII, the frequency of which can change within wide limits) are considerably smaller than the frequencies of the links of the stabilization loop, with the exception of link II.

As a result of the smallness of the coefficient $\dfrac{M_y^{\delta_э}}{J_y}$, link VII transmits considerably weakened signals and in practical calculations it can be considered that $W_7(p) = 0$.

Let us examine more closely the links which form the stabilization loop. In analyzing their characteristics we shall pay special attention to the changes due to their dependence on the air density and flight velocity. Here, just as in Section 2, we shall disregard the influence of air compressibility (Mach number M) on the force and moment coefficients, or in other words, we shall consider the change of forces and moments only with the change of the dynamic pressure

$$q = \frac{\rho V^2}{2}$$

The transfer function of the oscillating link I can be written in an expanded form:

$$W_1(p) = \frac{1}{p^2 - \left(c_z^\beta \dfrac{S}{2m}\rho V + m_y^{\omega_y}\dfrac{l^2 S}{4J_y}\rho V\right)p + c_z^\beta \dfrac{m_y^{\omega_y} S^2 l^2}{4J_y\, 2m}\rho^2 V^2 - m_y^\beta \dfrac{Sl}{2J_y}\rho V^2} =$$

$$= \frac{1}{p^2 + 2\zeta_1\omega_1 p + \omega_1^2}.$$

The reference frequency of this link equals

$$\omega_1 = \frac{1}{T_1} = \frac{\rho S V}{2m}\sqrt{c_z^\beta \frac{\bar{m}_y^{\omega_y}}{2\bar{r}_y^2} - \frac{m_y^\beta}{\bar{r}_y^2}\frac{2m}{\rho S l}}.$$

From this formula it follows that the frequency ω_1 is proportional to flight velocity V. With increasing altitude ω_1 decreases. Usually the value $c_z^\beta \dfrac{\bar{m}_y^{\omega_y}}{2\bar{r}_y^2}$ is small in comparison with $\dfrac{m_y^\beta}{\bar{r}_y^2}\dfrac{2m}{\rho S l}$ for all altitudes; therefore we can approximately assume that

$$\omega_1 = \frac{\rho S V}{2m}\sqrt{\frac{2m}{\rho S l}}\sqrt{\frac{m_y^\beta}{\bar{r}_y^2}} = \frac{\sqrt{\mu}}{\tau}\sqrt{\frac{m_y^\beta}{\bar{r}_y^2}},$$

where

$$\mu = \frac{2m}{\rho S l}, \quad \tau = \frac{2m}{\rho S V}.$$

From this approximate expression for ω_1 it follows that the frequency ω_1 decreases with increasing altitude approximately as $\sqrt{\rho}$. The relative damping coefficient of the link under consideration ζ_1 can be presented in the following form:

$$\zeta_1 = \frac{\frac{\rho S V}{2m}\left(c_z^\beta + \frac{\bar{m}_y^{\omega_y}}{2\bar{r}_y^2}\right)}{\frac{\rho S V}{2m}\sqrt{-\frac{m_y^\beta}{\bar{r}_y^2}\frac{2m}{\rho S l} + c_z^\beta \frac{\bar{m}_y^{\omega_y}}{2\bar{r}_y^2}}} = \frac{c_z^\beta + \frac{\bar{m}_y^{\omega_y}}{2\bar{r}_y^2}}{\sqrt{-\mu\frac{m_y^\beta}{\bar{r}_y^2} + c_z^\beta \frac{\bar{m}_y^{\omega_y}}{2\bar{r}_y^2}}}$$

As can be seen from this formula, the relative damping coefficient of this link does not depend on the flight velocity. With increase of altitude, damping deteriorates as a result of the increase of the relative density μ. Here, just as in the expression for ω_1, we can neglect the quantity

$$c_z^\beta \frac{\bar{m}_y^{\omega_y}}{2\bar{r}_y^2}.$$

Then the relative damping coefficient can be written approximately as

$$\zeta_1 = \frac{c_z^\beta + \frac{\bar{m}_y^{\omega_y}}{2\bar{r}_y^2}}{\sqrt{\frac{2m}{\rho S l}}\sqrt{-m_y^\beta}}.$$

Therefore, the relative damping coefficient of link I, decrease with ncrease of altitude, just as the frequency ω_1, approximately proportionately to $\sqrt{\rho}$.

The amplification factor of this link is

$$k_1 = \frac{1}{\left(c_z^\beta \frac{\bar{m}_y^{\omega_y}}{2\bar{r}_y^2} - \frac{m_y^\beta}{\bar{r}_y^2}\mu\right)} \frac{1}{\tau^2}.$$

The differentiating link II with transfer function

$$W_2(p) = V\frac{S\rho V}{2m}\left[\left(-\frac{m_x^\beta}{\bar{r}_x^2} + \frac{m_y^\beta}{\bar{r}_y^2}\operatorname{tg}\vartheta\right)p + \left(\frac{m_x^\beta}{\bar{r}_x^2 l}\frac{\bar{m}_y^{\omega_y}}{2\bar{r}_y^2} - \frac{m_y^\beta}{\bar{r}_y^2 l}\frac{\bar{m}_x^{\omega_y}}{2\bar{r}_x^2}\right)\frac{S\rho V}{2m} + \right.$$
$$\left. + \operatorname{tg}\vartheta\left(\frac{m_x^\beta}{\bar{r}_x^2 l}\frac{\bar{m}_y^{\omega_x}}{2\bar{r}_y^2} - \frac{m_y^\beta}{\bar{r}_y^2 l}\frac{\bar{m}_x^{\omega_x}}{2\bar{r}_x^2}\right)\frac{S\rho V}{2m}\right]$$

has the reference frequency

$$\omega_2 = \frac{1}{T_2} = \frac{S\rho V}{2m\cdot 2\bar{r}_y^2}\left|\left(\bar{m}_y^{\omega_y} - \frac{m_y^\beta \bar{m}_x^{\omega_y}}{m_x^\beta}\right) + \operatorname{tg}\vartheta\left(\bar{m}_y^{\omega_x} - \frac{m_y^\beta \bar{m}_x^{\omega_x}}{m_x^\beta}\right)\right| =$$
$$= \frac{1}{2\bar{r}_y^2\tau}\left|\left(\bar{m}_y^{\omega_y} - \frac{m_y^\beta \bar{m}_x^{\omega_y}}{m_x^\beta}\right) + \operatorname{tg}\vartheta\left(\bar{m}_y^{\omega_x} - \frac{m_y^\beta \bar{m}_x^{\omega_x}}{m_x^\beta}\right)\right| =$$
$$= \left|\frac{a_{4,0}}{-\frac{M_x^\beta}{J_x} + \frac{M_y^\beta}{J_y}\operatorname{tg}\vartheta}\right|.$$

From this it can be seen that the frequency ω_2 is proportional to fligh velocity and air density.

Since the coefficient $a_{4,0}$ for contemporary aircraft is small, the freqiency ω_2 for all altitudes and velocities is considerably smaller then the frequency of the other links which form the stabilization loop. It is proportional to the flight velocity and air density.

The amplification factor of link II can be expressed in the following form:

$$k_2 = \frac{V}{\tau^2}\left(\frac{m_x^\beta}{\bar{r}_x^2 l}\,\frac{m_y^{\bar{\omega}_y}}{2\bar{r}_y^2} - \frac{m_y^\beta}{\bar{r}_y^2 l}\,\frac{m_x^{\bar{\omega}_y}}{2\bar{r}_x^2}\right) + \operatorname{tg}\vartheta\left(\frac{m_x^\beta}{\bar{r}_x^2 l}\,\frac{m_y^{\bar{\omega}_x}}{2\bar{r}_y^2} - \frac{m_y^\beta}{\bar{r}_y^2 l}\,\frac{m_x^{\bar{\omega}_x}}{2\bar{r}_x^2}\right).$$

Link III has the transfer function

$$W_3'(p) = \frac{1}{V\,\dfrac{S\rho V}{2m\bar{r}_y^2 l}\left[-m_y^\beta p + m_y^\beta m_x^{\bar{\omega}_x}\,\dfrac{S\rho V}{2m\bar{r}_x^2} - m_x^\beta m_y^{\bar{\omega}_x}\,\dfrac{S\rho V}{2m\bar{r}_x^2}\right]}.$$

Its reference frequency is

$$\omega_3 = \frac{1}{T_3} = \frac{S\rho V}{2m\,2\bar{r}_x^2}\left(-m_x^{\bar{\omega}_x} + \frac{m_x^\beta m_y^{\bar{\omega}_x}}{m_y^\beta}\right) = \frac{1}{2\tau\bar{r}_x^2}\left(-m_x^{\bar{\omega}_x} + \frac{m_x^\beta m_y^{\bar{\omega}_x}}{m_y^\beta}\right).$$

The frequency ω_3, as also the frequency ω_2, is proportional to air density and flight velocity.

The amplification factor to link III is

$$k_3 = \frac{2\tau^2\bar{r}_y^2\bar{r}_x^2 l}{V\left(m_y^\beta m_x^{\bar{\omega}_x} - m_x^\beta m_y^{\bar{\omega}_x}\right)}.$$

The differentiating link V of second order with transfer function

$$W_5(p) = \frac{\rho S V}{2m\bar{r}_y^2}\left(\frac{m_y^{\bar{\omega}_x}}{2}p^2 - \frac{m_y^{\bar{\omega}_x} c_z^\beta}{2}\,\frac{\rho S V}{2m}\,p + \frac{g}{l}\,m_y^\beta\right)$$

has a reference frequency

$$\omega_5 = \sqrt{\frac{2g}{l}}\sqrt{\left|\frac{m_y^\beta}{m_y^{\bar{\omega}_x}}\right|}.$$

The amplification factor of this link is

$$k_5 = \frac{1}{\tau\bar{r}_y^2}\,\frac{g}{l}\,m_y^\beta.$$

The relative damping coefficient ζ_5 can be written in the following form:

$$\zeta_5 = -c_z^\beta\,\frac{\rho V S}{2m}\sqrt{\frac{l}{2g}}\sqrt{\left|\frac{m_y^{\bar{\omega}_x}}{m_y^\beta}\right|} = -\frac{1}{\omega_5\tau}\,c_z^\beta.$$

As can be seen from these formulas, the reference frequency of link V does not depend on air density and flight velocity.

The coefficient ζ_5 increases with increasing velocity, and decreases with increasing altitude, i.e., it deteriorates with increasing altitude. It is very small, and the link is practically a harmonic link.

In old airplanes with small flight velocity, the frequencies of all the links I – III were small in comparison with the frequency of link V. Therefore, for old airplanes link V could be considered to be purely amplifying with the amplification factor

$$k_5 = \frac{g}{V} \frac{M_y^\beta}{J_y} \cos\theta.$$

In other words, for old airplanes we could disregard the mixed derivative $M_y^\omega x$. For contemporary airplanes, as a result of their higher velocity, the frequencies of links I and III are comparable with the frequency of link V. Therefore, this link can not be considered now as being purely amplifying. This fact explains the instruction, frequently encountered in the literature, that for contemporary airplanes it is necessary to consider the angle between the principal axis of inertia and the velocity vector, because this angle influences the value of $M_y^\omega x$ (see Appendix II) [2].

Now let us consider the control links. The transfer function of link VI can be expressed in following form:

$$W_6(p) = m_y^{\delta_H} \frac{Sl}{2J_y} \rho V^2 p + \frac{Sl}{2J_y} \rho V^2 \frac{S\rho V}{2m} (m_y^\beta c_z^{\delta_H} - m_y^{\delta_H} c_z^\beta) =$$
$$= \frac{\mu}{r_y^2 \tau^2} \left[m_y^{\delta_H} p + (m_y^\beta c_z^{\delta_H} - m_y^{\delta_H} c_z^\beta) \frac{1}{\tau} \right].$$

Since the value of $m_y^{\delta_H} c_z^\beta$ is usually larger than the value of $m_y^\beta c_z^{\delta_H}$, this link is a stable differentiating link. Normally the difference $m_y^\beta c_z^{\delta_H} - m_y^{\delta_H} c_z^\beta$ is small, and therefore the reference frequency of link IV is also small:

$$\frac{1}{T_6} = \omega_6 = \frac{1}{\tau} \frac{m_y^\beta c_z^{\delta_H} - m_y^{\delta_H} c_z^\beta}{m_y^{\delta_H}}$$

As can be seen from this expression, the reference frequency of link VI decreases, with increase of altitude, proportionately to the density ρ and increases proportionately to the velocity V.

The amplification factor of link VI is

$$k_6 = \frac{\mu}{r_y^2 \tau^3} (m_y^\beta c_z^{\delta_H} - m_y^{\delta_H} c_z^\beta).$$

From this relation it can be seen that k_6 decreases as ρ^2 with increase in altitude of flight, and increases proportionately to the cube of velocity, V^3.

The transfer function of link VII can be written as

$$W_7(p) = m_y^{\delta_э} \frac{Sl}{2J_y} \rho V^2 \left(p - c_z^\beta \frac{S\rho V}{2m} \right) = m_y^{\delta_э} \frac{\mu}{r_y^2 \tau^2} \left(p - \frac{c_z^\beta}{\tau} \right).$$

This expression shows that the reference frequency of link VII is

$$\omega_7 = -c_z^\beta \frac{1}{\tau}.$$

Therefore, with increase of flight velocity ω_7 decreases proportionately to the density ρ and increases proportionately to flight velocity V. The amplification factor of link VII is

$$k_7 = -m_y^{\delta_э} c_z^\beta \frac{\mu}{r_y^2 \tau^2}.$$

This coefficient decreases proportionately to ρ and increases proportionately to V^3.

In contemporary airplanes, usually, the value of $m_y^{\delta_э}$ is close to zero. Therefore (as in the above example) we can disregard link VII. It is possible that we will have to consider it at very high flight velocities in the case of ailerons for which the quantity $m_y^{\delta_э} \neq 0$.

The transfer function of the differentiating link VII of second order can be written as

$$W_8(p) = \sin\alpha p^2 + \left(-\bar{m}_y^{\omega_y}\frac{l^2 S\rho V}{4J_y}\sin\alpha + \bar{m}_y^{\omega_x}\frac{l^2 S\rho V}{4J_y}\cos\alpha + \frac{g}{V}\cos\vartheta\right)p -$$

$$-\frac{g}{V}\cos\vartheta\left(\bar{m}_y^{\omega_y}\frac{l^2 S\rho V}{4J_y} + \bar{m}_y^{\omega_x}\frac{l^2 S\rho V}{4J_y}\operatorname{tg}\vartheta\right) =$$

$$= \sin\alpha p^2 + \left[\frac{1}{2\tau\bar{r}_y^2}\left(-\bar{m}_y^{\omega_y}\sin\alpha + \bar{m}_y^{\omega_x}\cos\alpha\right) + \frac{g}{V}\cos\vartheta\right]p -$$

$$-\frac{g}{V}\frac{\cos\vartheta}{2\tau\bar{r}_y^2}\left(\bar{m}_y^{\omega_y} + \bar{m}_y^{\omega_x}\operatorname{tg}\vartheta\right).$$

When $\alpha = 0$, one root of the polynomial $W_8(p)$ becomes ∞. As a result, link VIII becomes a differentiating link of first order. In this case the transfer function has the form

$$W_8(p) = \left(\frac{1}{2\tau\bar{r}_y^2}\bar{m}_y^{\omega_x} + \frac{g}{V}\cos\vartheta\right)p - \frac{g}{V}\frac{\cos\vartheta}{2\tau\bar{r}_y^2}\left(\bar{m}_y^{\omega_y} + \bar{m}_y^{\omega_x}\operatorname{tg}\vartheta\right).$$

The reference frequency of link VIII in this case is

$$\omega_8 = \frac{1}{T_8} = \left|\frac{g}{V}\frac{\cos\vartheta}{2\tau\bar{r}_y^2}\frac{\bar{m}_y^{\omega_y} + \bar{m}_y^{\omega_x}\operatorname{tg}\vartheta}{\frac{1}{2\tau\bar{r}_y^2}\bar{m}_y^{\omega_x} + \frac{g}{V}\cos\vartheta}\right| =$$

$$= \frac{g\rho S}{4m\bar{r}_y^2}\left|\frac{\bar{m}_y^{\omega_y}\cos\vartheta + \bar{m}_y^{\omega_x}\sin\vartheta}{\frac{g}{V}\cos\vartheta + \frac{\rho S V}{4m\bar{r}_y^2}\bar{m}_y^{\omega_x}}\right|.$$

From this formula it follows that the reference frequency of the link depends on density ρ and velocity V in a complicated manner. For example, if in the denominator we can neglect the second term $\frac{\rho S V}{4m\bar{r}_y^2}\bar{m}_y^{\omega_x}$, then the reference frequency varies proportionately to ρV. On the other hand, if we could disregard the first member (at very high velocities), the reference frequency varies in inverse proportion to velocity and does not depend upon ρ.

In the case, when the angle of attack α is not zero, but is small, the transfer function $W_8(p)$ has two real roots and link VIII can be expressed in the form of two links of second order in tandem. In this case, the roots of the transfer function can be approximately expressed by the formulas

$$p_1 = -\left[\frac{1}{2\tau V\bar{r}_y^2}\bar{m}_y^{\omega_x}\cos\alpha + \frac{g}{V}\cos\vartheta\right]\frac{1}{\sin\alpha},$$

$$p_2 = \frac{g}{2\tau V\bar{r}_y^2}\frac{\bar{m}_y^{\omega_y}\cos\vartheta + \bar{m}_y^{\omega_x}\sin\vartheta}{\frac{1}{2\tau\bar{r}_y^2}\bar{m}_y^{\omega_x} + \frac{g}{V}\cos\vartheta}$$

and the transfer functions of the two tandem-connected links will be

$$W_8'(p) = \sin\alpha p + \frac{1}{2\tau V\bar{r}_y^2}\bar{m}_y^{\omega_x}\cos\alpha + \frac{g}{V}\cos\vartheta,$$

$$W_8''(p) = p - \frac{g}{2\tau V\bar{r}_y^2}\frac{\bar{m}_y^{\omega_y}\cos\vartheta + \bar{m}_y^{\omega_x}\sin\vartheta}{\frac{1}{2\tau\bar{r}_y^2}\bar{m}_y^{\omega_x} + \frac{g}{V}\cos\vartheta}.$$

Until the reference frequency of link VIII', which equals

$$\omega_8' = -p_1 = \frac{1}{\sin\alpha}\left[\frac{1}{2\tau \bar{r}_y^2} m_y^{\bar{\omega}_x} \cos\alpha + \frac{g}{V} \cos\vartheta\right],$$

is large (as a result of the smallness of sin α) link VIII' can be considered to be purely amplifying with amplification factor

$$k_8' = \frac{1}{2\tau V \bar{r}_y^2} m_y^{\bar{\omega}_x} + \frac{g}{V} \cos\vartheta.$$

In this case the sequence of links VIII' and VIII'' is practically equivalent to a single link with the transfer function $W_8(p)$, which is obtained at $\alpha = 0$.

With increase of the attack angle the frequency of link VIII' decreases and can become commensurate with the frequency of link VIII. In this case it is necessary to conduct an analysis of the transfer function $W_8(p)$ with actual figures.

Let us note that the angle α can reach large values. Let us also remind ourselves that we are reckoning the angle of attack from the principal axis of inertia (see Section 4). When the principal axis of inertia is deflected downwards considerably (such deflection in particular cases can reach a magnitude of the order of 15-20°), and the angle of attack relative to the longitudinal axis (or the wing chord) has a value of the order of 10°, then the angle of attack relative to the principal axis of inertia can reach 25-30°.

The amplification factor of link VIII in all cases is

$$k_8 = -\frac{g}{2\tau V r_y^2}\left(m_y^{\bar{\omega}_y} \cos\vartheta + m_y^{\bar{\omega}_x} \sin\vartheta\right) = \frac{g\rho S}{4m\bar{r}_y^2}\left(-m_y^{\bar{\omega}_y} \cos\vartheta - m_y^{\omega_x} \sin\vartheta\right);$$

it does not depend on the velocity $\underline{V}$ and varies proportionately to the density ρ.

Even the brief analysis presented above shows that the transfer function $W_8(p)$ depends on the flight regime, aerodynamic parameters, and design parameters (for example, the angle of inclination of the principal axis of inertia) in a very complicated way. This fact explains the complicated change of the angle of sideslip during transient processes, because it is the transfer function $W_8(p)$ which determines β.

Section 6. Transfer Functions of Control. Simplification of Equations of Motion and Transfer Functions.

The stabilization loop (system of links I – V) can be conveniently opened after link V. The transfer function of the opened loop has the form

$$W(p) = -W_1(p)\, W_2(p)\, W_3(p)\, W_4(p)\, W_5(p).$$

The characteristic equation for an airplane with fixed controls will be

$$\frac{p}{W_1(p)\, W_3(p)} - W_2(p)\, W_6(p) = \frac{p}{W_1(p)\, W_2(p)} [1 + W(p)] = 0.$$

It is easy to confirm that this equation agrees with the usual equation obtained by classical methods. In particular, the term not containing $\underline{p}$ in the characteristic equation is

$$a_4 = \frac{g}{V} \cos\vartheta \cdot a_{4,0}.$$

Therefore, a necessary condition for stability (condition of absence of aperiodic instability), will be the usual condition

$$a_{4,0} > 0,$$

that is, the condition $k_2 < 0$, where k_2 is the amplification factor of link II.

Having the transfer function of an opened system, we can apply well-known frequency methods [3] to the analysis of stability.

The equation of motion (4.2) can be written in abbreviated form in the following manner:

$$\left.\begin{aligned} \omega_y &= W_1(p)\,[W_5(p)\,\gamma + W_7(p)\,\delta_{\text{Э}} + W_6(p)\,\delta_H], \\ p\gamma &= W_3(p)\left[W_2(p)\,\omega_y - \left(\frac{M_x^{\delta_{\text{Э}}}}{J_x}\frac{M_y^{\beta}}{J_y} - \frac{M_y^{\delta_{\text{Э}}}}{J_y}\frac{M_x^{\beta}}{J_x}\right)\delta_{\text{Э}} - \left(\frac{M_x^{\delta_H}}{J_x}\frac{M_y^{\beta}}{J_y} - \frac{M_y^{\delta_H}}{J_y}\frac{M_x^{\beta}}{J_x}\right)\delta_H\right]. \end{aligned}\right\} \quad (6.1)$$

From this, by solving the equation for γ and ω_y, we obtain

$$\left.\begin{aligned} p\gamma &= W_3(p)\,\frac{W_1(p)\,W_2(p)\,W_6(p) - \left(\frac{M_x^{\delta_H}}{J_x}\frac{M_y^{\beta}}{J_y} - \frac{M_y^{\delta_H}}{J_y}\frac{M_x^{\beta}}{J_x}\right)}{1+W(p)}\,\delta_H + \\ &\quad + W_3(p)\,\frac{W_1(p)\,W_2(p)\,W_7(p) - \left(\frac{M_x^{\delta_{\text{Э}}}}{J_x}\frac{M_y^{\beta}}{J_y} - \frac{M_y^{\delta_{\text{Э}}}}{J_y}\frac{M_x^{\beta}}{J_x}\right)}{1+W(p)}\,\delta_{\text{Э}}; \\ \omega_y &= W_1(p)\,\frac{W_6(p) - W_3(p)\,W_4(p)\,W_5(p)\left(\frac{M_x^{\delta_H}}{J_x}\frac{M_y^{\beta}}{J_y} - \frac{M_y^{\delta_H}}{J_y}\frac{M_x^{\beta}}{J_x}\right)}{1+W(p)}\,\delta_H + \\ &\quad + W_1(p)\,\frac{W_7(p) - W_3(p)\,W_4(p)\,W_5(p)\left(\frac{M_x^{\delta_{\text{Э}}}}{J_x}\frac{M_y^{\beta}}{J_y} - \frac{M_y^{\delta_{\text{Э}}}}{J_y}\frac{M_x^{\beta}}{J_x}\right)}{1+W(p)}\,\delta_{\text{Э}}. \end{aligned}\right\} \quad (6.2)$$

Formulas (6.2) give the transfer functions relating γ and ω_y to δ_H and $\delta_{\text{Э}}$. Having the frequency characteristics of the links, we can easily construct the frequency characteristics of the control system, by using the well-known diagram [3] which allows us to construct the frequency characteristic of a closed system, when the frequency characteristic of the open system is known. However, here we lose clarity and simplicity. Therefore, a simplified theory of the transfer functions of the control system will be given later, based on the fact that frequencies of the individual links are very far apart (Section 5).

In Section 5, we have seen that the reference frequency of link II is usually considerably lower than the frequency of the rest of the links. Therefore, in the slow motion we can consider links I, III and V as purely amplifying; on this basis we can write the transfer functions as

$$W_1(p) = k_1 = \frac{1}{\frac{Z^{\beta}}{mV}\left(\frac{M_y^{\omega_y}}{J_y} + \frac{M_y^{\omega_x}}{J_y}\,\mathrm{tg}\,\vartheta\right) - \frac{M_y^{\beta}}{J_y}\frac{\cos\theta}{\cos\vartheta}},$$

$$W_2(p) = -\left(\frac{M_x^{\beta}}{J_x} - \frac{M_y^{\beta}}{J_y}\,\mathrm{tg}\,\vartheta\right)p - a_{4,0},$$

$$W_3(p) = k_3 = \frac{1}{\frac{M_y^{\beta}}{J_y}\frac{M_x^{\omega_x}}{J_x} - \frac{M_x^{\beta}}{J_x}\frac{M_y^{\omega_x}}{J_y}},$$

$$W_4(p) = \frac{1}{p},$$

$$W_5(p) = k_5 = \frac{g}{V}\frac{M_y^{\beta}}{J_y}\cos\vartheta.$$

Now the transfer function of the open system will be

$$W(p) = -k_1 k_3 k_5 W_2(p) \frac{1}{p},$$

$$1 + W(p) = \frac{-k_1 k_3 k_5 W_2(p) + p}{p}.$$

From this we obtain (6.2)

$$\left.\begin{aligned}
\gamma &= k_3 \frac{k_1 W_2(p) W_6(p) - \left(\frac{M_x^{\delta_H}}{J_x}\frac{M_y^{\beta}}{J_y} - \frac{M_y^{\delta_H}}{J_y}\frac{M_x^{\beta}}{J_x}\right)}{p - k_1 k_3 k_5 W_2(p)} \delta_H + \\
&+ k_3 \frac{k_1 W_2(p) W_7(p) - \left(\frac{M_x^{\delta_Э}}{J_x}\frac{M_y^{\beta}}{J_y} - \frac{M_y^{\delta_Э}}{J_y}\frac{M_x^{\beta}}{J_x}\right)}{p - k_1 k_3 k_5 W_2(p)} \delta_Э; \\
\omega_y &= k_1 \frac{p W_6(p) - k_3 k_5 \left(\frac{M_x^{\delta_H}}{J_x}\frac{M_y^{\beta}}{J_y} - \frac{M_y^{\delta_H}}{J_y}\frac{M_x^{\beta}}{J_x}\right)}{p - k_1 k_3 k_5 W_2(p)} \delta_H + \\
&+ k_1 \frac{p W_7(p) - k_3 k_5 \left(\frac{M_x^{\delta_Э}}{J_x}\frac{M_y^{\beta}}{J_y} - \frac{M_y^{\delta_Э}}{J_y}\frac{M_x^{\beta}}{J_x}\right)}{p - k_1 k_3 k_5 W_2(p)} \delta_Э.
\end{aligned}\right\} \quad (6.3)$$

The equations of motion (6.1) will take the form

$$\left.\begin{aligned}
\omega_y &= k_1 k_5 \gamma + k_1 W_7(p) \delta_Э + k_1 W_6(p) \delta_H, \\
p\gamma &= k_3 W_2(p)\, \omega_y - k_3 \left(\frac{M_x^{\delta_Э}}{J_x}\frac{M_y^{\beta}}{J_y} - \frac{M_y^{\delta_Э}}{J_y}\frac{M_x^{\beta}}{J_x}\right)\delta_Э - k_3 \left(\frac{M_x^{\delta_H}}{J_x}\frac{M_y^{\beta}}{J_y} - \frac{M_y^{\delta_H}}{J_y}\frac{M_x^{\beta}}{J_x}\right)\delta_H.
\end{aligned}\right\} \quad (6.4)$$

Equations (6.3), as also Equations (6.4), show that in the slow motion the airplane can be regarded as an object of first order. This fact, which has been known before, receives a firm theoretical basis in this work. The theory allows us to calculate very easily the time constant of such an object. It equals

$$T_м = -\frac{1}{k_1 k_3 k_5 a_{4,0}} + \frac{\frac{M_x^{\beta}}{J_x} - \frac{M_y^{\beta}}{J_y} \operatorname{tg} \theta}{a_{4,0}}.$$

We do not cite a complete expanded expression for T_m, since it is very complex. For our example (H = 12 km) the time constant T_m is very large ($T_m = -7,300$ sec) and the airplane is practically neutral.

For the same aerodynamic data given in Section 5 (H = 12 km), we will obtain

$$\gamma = -0{,}615 \frac{p^2 + 0{,}01725p - 1{,}53}{p - 0{,}000137}\delta_H - \frac{3{,}56}{p - 0{,}000137}\delta_Э,$$

$$\omega_y = -0{,}327 \frac{p^2 + 0{,}0175p + 0{,}1148}{p - 0{,}000137}\delta_H + \frac{0{,}157}{p - 0{,}000137}\delta_Э.$$

As this example shows, the quadratic polynomials in the numerator can be practically considered as being constant at low frequencies. In other words, in Formula (6.1) we can consider

$$W_6(p) = W_7(p) = 0.$$

Finally, we have

$$\gamma = \frac{\omega_y}{k_1 k_5} = -\frac{k_3\left(\frac{M_x^{\delta_H}}{J_x}\frac{M_y^{\beta}}{J_y} - \frac{M_y^{\delta_H}}{J_y}\frac{M_x^{\beta}}{J_x}\right)}{p - k_1 k_3 k_5 W_2(p)}\delta_H - \frac{k_3\left(\frac{M_x^{\delta_Э}}{J_x}\frac{M_y^{\beta}}{J_y} - \frac{M_y^{\delta_Э}}{J_y}\frac{M_x^{\beta}}{J_x}\right)}{p - k_1 k_3 k_5 W_2(p)}\delta_Э \quad (6.3')$$

It is easily seen that Formulas (6.3') will be obtained if in the last two equations (moment equations) of the system of equations of motion (4.1) it is assumed that $p \equiv 0$ (that is, we assume that the moments are statically balanced), and in the force equation we disregard the term $\rho \sin \alpha \gamma$.

Most contemporary airplanes are nearly aperiodically neutral; for a neutral airplane $\alpha_{4,0} = 0$ and the transfer function $W_2(p)$ equals

$$W_2(p) = -\left(\frac{M_x^\beta}{J_x} - \frac{M_y^\beta}{J_y}\operatorname{tg}\vartheta\right)p.$$

Therefore, Equations (6.3) will assume the form

$$\left.\begin{aligned}
\gamma &= k_3\frac{-k_1W_6(p)\left(\frac{M_x^\beta}{J_x} - \frac{M_y^\beta}{J_y}\operatorname{tg}\vartheta\right)p - \left(\frac{M_x^{\delta_H}}{J_x}\frac{M_y^\beta}{J_y} - \frac{M_y^{\delta_H}}{J_y}\frac{M_x^\beta}{J_x}\right)}{1+k_1k_3k_5\left(\frac{M_x^\beta}{J_x} - \frac{M_y^\beta}{J_y}\operatorname{tg}\vartheta\right)}\frac{1}{p}\delta_H + \\
&+ k_3\frac{-k_1W_7(p)\left(\frac{M_x^\beta}{J_x} - \frac{M_y^\beta}{J_y}\operatorname{tg}\vartheta\right)p - \left(\frac{M_x^{\delta_Э}}{J_x}\frac{M_y^\beta}{J_y} - \frac{M_y^{\delta_Э}}{J_y}\frac{M_x^\beta}{J_x}\right)}{1+k_1k_3k_5\left(\frac{M_x^\beta}{J_x} - \frac{M_y^\beta}{J_y}\operatorname{tg}\vartheta\right)}\frac{1}{p}\delta_Э, \\
\omega_y &= k_1\frac{pW_6(p) - k_3k_5\left(\frac{M_x^{\delta_H}}{J_x}\frac{M_y^\beta}{J_y} - \frac{M_y^{\delta_H}}{J_y}\frac{M_x^\beta}{J_x}\right)}{1+k_1k_3k_5\left(\frac{M_x^\beta}{J_x} - \frac{M_y^\beta}{J_y}\operatorname{tg}\vartheta\right)}\frac{1}{p}\delta_H + \\
&+ k_1\frac{pW_7(p) - k_3k_5\left(\frac{M_x^{\delta_Э}}{J_x}\frac{M_y^\beta}{J_y} - \frac{M_y^{\delta_Э}}{J_y}\frac{M_x^\beta}{J_x}\right)}{1+k_1k_3k_5\left(\frac{M_x^\beta}{J_x} - \frac{M_y^\beta}{J_y}\operatorname{tg}\vartheta\right)}\frac{1}{p}\delta_Э.
\end{aligned}\right\} \quad (6.5)$$

In these formulas we can also disregard the transfer functions $W_6(p)$ and $W_7(p)$ in the slow motion; finally the formula for a neutral airplane assumes the simple form:

$$\gamma = \frac{\omega_y}{k_1k_5} = -\frac{k_3\left(\frac{M_x^{\delta_H}}{J_x}\frac{M_y^\beta}{J_y} - \frac{M_y^{\delta_H}}{J_y}\frac{M_x^\beta}{J_x}\right)}{1+k_1k_3k_5\left(\frac{M_x^\beta}{J_x} - \frac{M_y^\beta}{J_y}\operatorname{tg}\vartheta\right)}\frac{1}{p}\delta_H - \frac{k_3\left(\frac{M_x^{\delta_Э}}{J_x}\frac{M_y^\beta}{J_y} - \frac{M_y^{\delta_Э}}{J_y}\frac{M_x^\beta}{J_x}\right)}{1+k_1k_3k_5\left(\frac{M_x^\beta}{J_x} - \frac{M_y^\beta}{J_y}\operatorname{tg}\vartheta\right)}\frac{1}{p}\delta_Э. \quad (6.6)$$

Formulas (6.6) show that in this case the airplane is an object of zero order, the derivatives of the basic kinematic parameters γ and ω_y being absent.

Let us note that according to Equation (6.3') in the case of the slow motion we can always consider

$$\omega_y = k_1k_5\cdot\gamma.$$

In the case of the rapid motion we can consider that link II is a purely differentiating link with the transfer function

$$W_2(p) = \left(-\frac{M_x^\beta}{J_x} + \frac{M_y^\beta}{J_y}\operatorname{tg}\vartheta\right)p.$$

The transfer function of the opened system will be

$$W(p)=\left(\frac{M_x^{\beta}}{J_x}-\frac{M_y^{\beta}}{J_y}\operatorname{tg}\vartheta\right)W_1(p)\,W_5(p)\,W_3(p).$$

The abbreviated equations of motion will have the form

$$\omega_y=W_1(p)\left[W_5(p)\,\gamma+W_7(p)\,\delta_э+W_6(p)\,\delta_H\right],$$

$$p\gamma=W_3(p)\left[-\left(\frac{M_x^{\beta}}{J_x}-\frac{M_y^{\beta}}{J_y}\operatorname{tg}\vartheta\right)p\omega_y-\left(\frac{M_x^{\delta_э}}{J_x}\frac{M_y^{\beta}}{J_y}-\frac{M_y^{\delta_э}}{J_y}\frac{M_x^{\beta}}{J_x}\right)\delta_э-\left(\frac{M_x^{\delta_H}}{J_x}\frac{M_y^{\beta}}{J_y}-\frac{M_y^{\delta_H}}{J_y}\frac{M_x^{\beta}}{J_x}\right)\delta_H\right].$$

The airplane in this case represents a system of third order.

The second of these equations can easily be obtained by writing the moment equation about the neutral axis of sideslip (Section 4, Equation (4.2)) and assuming $\alpha_{4,0}=0$.

The approximate characteristic equation for the rapid motion can be written in the following form:

$$\frac{1}{W_1(p)\,W_3(p)}+\left(\frac{M_x^{\beta}}{J_x}-\frac{M_y^{\beta}}{J_y}\operatorname{tg}\vartheta\right)W_5(p)=\frac{p}{W_1(p)\,W_3(p)}\left[1+W(p)\right]=0.$$

As the calculations at the end of this section (Table V) show, the roots of this characteristic equation are, practically, almost identical to the roots of the polynomials $\frac{1}{W_1(p)}$ and $\frac{1}{W_3(p)}$; this is because the amplification factor of link V is small in comparison with the product of the amplification factor $\frac{1}{k_1k_3}$; as the velocity increases the difference will also increase.

In other words, for the case of the rapid motion we can approximately consider (see the analogous discussion in Section 3)

$$1+W(p)\approx 1.$$

Therefore the transfer function (6.2) can be rewritten in a simpler form:

$$\left.\begin{aligned}
p\gamma={}&W_1(p)\,W_3(p)\left[\left(-\frac{M_x^{\beta}}{J_x}+\frac{M_y^{\beta}}{J_y}\operatorname{tg}\vartheta\right)p\,W_6(p)-\left(\frac{M_x^{\delta_H}}{J_x}\frac{M_y^{\beta}}{J_y}-\frac{M_y^{\delta_H}}{J_y}\frac{M_x^{\beta}}{J_x}\right)\frac{1}{W_1(p)}\right]\delta_H+\\
&+W_1(p)\,W_3(p)\left[\left(-\frac{M_x^{\beta}}{J_x}+\frac{M_y^{\beta}}{J_y}\operatorname{tg}\vartheta\right)p\,W_7(p)-\left(\frac{M_x^{\delta_э}}{J_x}\frac{M_y^{\beta}}{J_y}-\frac{M_y^{\delta_э}}{J_y}\frac{M_x^{\beta}}{J_x}\right)\frac{1}{W_1(p)}\right]\delta_э,\\
p\omega_y={}&W_1(p)\,W_3(p)\left[\frac{pW_6(p)}{W_3(p)}-\left(\frac{M_x^{\delta_H}}{J_x}\frac{M_y^{\beta}}{J_y}-\frac{M_y^{\delta_H}}{J_y}\frac{M_x^{\beta}}{J_x}\right)W_5(p)\right]\delta_H+\\
&+W_1(p)\,W_3(p)\left[\frac{pW_7(p)}{W_3(p)}-\left(\frac{M_x^{\delta_э}}{J_x}\frac{M_y^{\beta}}{J_y}-\frac{M_y^{\delta_э}}{J_y}\frac{M_x^{\beta}}{J_x}\right)W_5(p)\right]\delta_э.
\end{aligned}\right\}\quad(6.7)$$

In many cases these formulas can be simplified even more, if we consider that for the rapid motion we can assume link VI to be purely differentiating (its reference frequency is small) with the transfer function

$$W_6(p)=\frac{M_y^{\delta_H}}{J_y}\,p.$$

Also, in the first square brackets of the second formula we can frequently assume $W_5(p)=0$ because of the smallness of amplificiation factor k_5. Finally, because of the smallness of the quantity $\frac{M_y^{\delta_э}}{J_y}$ we can assume $W_7(p)\equiv 0$.

We now obtain simple formulas for the rapid motion:

$$\left.\begin{aligned} p\gamma = W_1(p)\,W_3(p)\Bigg[\left(-\frac{M_x^\beta}{J_x}+\frac{M_y^\beta}{J_y}\operatorname{tg}\vartheta\right)\frac{M_y^{\delta_H}}{J_y}\,p^2-\left(\frac{M_x^{\delta_H}}{J_x}\,\frac{M_y^\beta}{J_y}-\frac{M_y^{\delta_H}}{J_y}\,\frac{M_x^\beta}{J_x}\right)\frac{1}{W_1(p)}\Bigg]\delta_H- \\ -\frac{M_x^{\delta_э}}{J_x}\,\frac{M_y^\beta}{J_y}\,W_3(p)\,\delta_э, \\ \omega_y = W_1(p)\,\frac{M_y^{\delta_H}}{J_y}\,p\delta_H-\frac{M_x^{\delta_э}}{J_x}\,\frac{M_y^\beta}{J_y}\,\frac{W_1(p)\,W_3(p)\,W_5(p)}{p}\,\delta_э. \end{aligned}\right\} \quad (6.8)$$

From these formulas it is easily seen that in the case of control only by the ailerons $(\delta_H = 0)$ we will obtain the usual expression for the angle of bank (equation of rolling motion)

$$\gamma = \frac{\dfrac{M_x^{\delta_э}}{J_x}}{p\left[p-\dfrac{M_x^{\omega_x}}{J_x}+\dfrac{M_y^{\omega_x}}{J_y}\,\dfrac{\dfrac{M_x^\beta}{J_x}}{\dfrac{M_y^\beta}{J_y}}\right]}\,\delta_э. \qquad (6.9)$$

The angular velocity of yaw in this case will be simply expressed in terms of the angle of bank

$$\omega_y = W_1(p)\;W\;(p)\,\gamma. \qquad (6.10)$$

Because the amplification factor k_5 is small, the angular velocity of yaw will be insignificant.

In the case of control only with the rudder $(\delta_э = 0)$ we will have

$$\omega_y = W_1(p)\,\frac{M_y^{\delta_H}}{J_y}\,p\delta_H, \qquad (6.11)$$

that is, the usual approximate expression of transfer function relating the angular velocity of yaw to the rudder deflection. For the angle of bank we will obtain the more complicated expression

$$p\gamma = W_1(p)\,W_3(p)\left[\left(-\frac{M_x^\beta}{J_x}+\frac{M_y^\beta}{J_y}\operatorname{tg}\vartheta\right)\frac{M_y^{\delta_H}}{J_y}\,p^2-\left(\frac{M_x^{\delta_H}}{J_x}\,\frac{M_y^\beta}{J_y}-\frac{M_y^{\delta_H}}{J_y}\,\frac{M_x^\beta}{J_x}\right)\frac{1}{W_1(p)}\right]\delta_H, \qquad (6.12)$$

which can be simplified only by using additional assumptions.

For our case (H = 12 km) we have the following numerical expression:

$$\gamma = -\,0{,}466\,\frac{p^2-0{,}505p-4{,}64}{p^2+0{,}249p+2{,}28}\,\frac{1}{p}\,\delta_H,$$

from which we can see that all the terms in the numerator are significant and in general cannot be disregarded.

From the above it follows that the three investigated types of motion represent a spiral motion (slow motion), an oscillating motion (rapid motion during steering with the rudder) and a rolling motion (rapid motion during steering with the ailerons).

For each of the types of motion the method presented, based on an analysis of the block diagram, allows one easily and simply to write down abbreviated equations of motion.

For the rolling and oscillating motions, in the denominators of the transfer functions we have the polynomials $\frac{1}{W_1(p)}$ and $\frac{1}{W_3(p)}$; therefore, by setting these polynomials equal to zero we should obtain approximate values for the roots of the complete, exact characteristic equation: such a calculation was made for our example (H = 12 km) and is shown in Table V.

The complete characteristic equation for this case has the following form:

$$p^4+1{,}909p^3+2{,}69p^2+3{,}95p-0{,}000549=0.$$

Calculations show that natural airplane motions are usually characterized by two real roots p_1 and p_2 (p_2 is the small root of the characteristic equation and p_1 is the large root and a pair of complex roots p_3 and p_4.

The root p_1 characterizes the rolling motion of the airplane. It should be compared with the root of the equation $\frac{1}{W_3(p)} = 0$. The natural frequency of link III for this case equals 1.608, i.e., within an accuracy of 5% it coincides with the exact root of the characteristic equation.

The pair of conjugate complex root $p_{3,4}$ characterizes the natural airplane motion and should be compared with the roots of the equation $\frac{1}{W_1(p)} = 0$.

The frequency corresponding to this pair of roots coincides with frequency ω_1 of link I with an accuracy of up to 1.6%, and the damping coefficient coincides with the damping coefficient of link I with an accuracy of up to 14%.

The small root p_2 approximately equals $p_2 \approx -\frac{1}{T_m}$; it can be calculated with a very high degree of accuracy.

TABLE V

Values of roots of characteristic equation.	Approximate values of roots
$p_1=-1{,}695$	$p_1=-\omega_3=-1{,}608$
$p_2=0{,}000139$	$p_2=-\frac{1}{T_M}=0{,}000136$
$p_{3,4}=-0{,}107\pm 1{,}525$	$p_{3,4}=-0{,}081\pm 1{,}52$*

In this section we have considered various simplifications for the case of a "typical" airplane, which is nearly aperiodically neutral. A similar analysis and corresponding simplifications can easily be made for an arbitrary case when the airplane characteristics are greatly different from the usual ones.

CONCLUSIONS

1. The system of differential equations of perturbed airplane motion (equations for the variations), for longitudinal as well as for lateral motion, can be expressed in the form of a simple single-loop block diagram with links of first and second order. The signals at the input and output of the links have a definite physical meaning.

2. The representation of a system of equations as a block diagram allows us to apply, in the investigation of stability and control problems, contemporary methods of general control theory: frequency methods of compensating net works, circuit analysis, etc.

3. The representation as a block diagram allows us to obtain simple approximate expressions for the transfer functions of the airplane control system depending on the frequency range of the various types of motion. In particular, in contrast to existing methods, which give us approximate transfer functions for rapid angular motions relative to the center of gravity, we can easily obtain approximate expressions for transfer functions for the slow motions, connected with displacements of the center of gravity.

4. The representation in the form of a block diagram allows us easily to formulate approximate differential equations for various practical cases.

* Roots of the polynomial $\frac{1}{W_1(p)}$.

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APPENDIX I

EQUATIONS OF PERTURBED LATERAL MOTION OF AN AIRPLANE

Because derivations of the equations of lateral motion are presented either in literature which is out of date [2] or in rare publications having a small circulation, a derivation of these equations is given in this appendix.

We will use two orthogonal systems of coordinate axes: an earth system, and a system rigidly connected with the airplane. The origin of both systems is located at the center of gravity of the airplane O (Fig. 13). The earth axis Oy_0 will be directed vertically upwards, the axes Ox_0 and Oz_0 – in the horizontal plane.

The airplane's system of axes will be chosen in such a way that the axis Oz will be directed perpendicularly to the symmetry plane, and to the right if we are looking in the direction of flight; the axes Ox and Oy will be located in the symmetry plane of the airplane (axis Ox forward, axis Oy upward).

We will specify the relative orientation of the two coordinate trihedrons Oxyz and $Ox_0y_0z_0$ by three angles. The yaw angle ψ will be the angle between the axis Ox_0 and the projection of axis Ox on the horizontal plane (straight line Ox'). The pitch angle ϑ will be the angle between axis Ox and the horizontal plane. The angle of bank γ will be the angle between the airplane symmetry plane and the vertical plane which passes through the axis Ox (in other words, the angle between the axis Oz and the straight line Oz').

These angles can also be represented in another form. Let us superimpose the airplane trihedron on the earth trihedron. Let us turn the airplane trihedron through the angle ψ about the vertical axis Oy_0. The axes Ox_0 and Oz_0 will assume the position Ox' and Oz'. Now let us turn the airplane about the new axis Oz' through the angle ϑ. The axis Ox will assume it final location. By turning the airplane about the logitudinal axis Ox through the angle of bank γ, we will obtain the final position.

Table VI shows the direction cosines for the transformation from the earth axes to the airplane axes.

TABLE VI

Axes	Ox_0	Oy_0	Oz_0
Ox	$\cos\vartheta\cos\psi$	$\sin\vartheta$	$-\cos\vartheta\sin\psi$
Oy	$\sin\gamma\sin\psi-\cos\gamma\cos\psi\sin\vartheta$	$\cos\gamma\cos\vartheta$	$\sin\gamma\cos\psi+\cos\gamma\sin\psi\sin\vartheta$
Oz	$\cos\gamma\sin\psi+\sin\gamma\cos\psi\sin\vartheta$	$-\sin\gamma\cos\vartheta$	$\cos\gamma\cos\psi-\sin\gamma\sin\psi\sin\vartheta$

From the above it is clear that the angular velocity vector can be considered as a geometric sum of three vectors (see Fig. 13): the angular velocity of rotation about the vertical axis Oy_0, the magnitude of which is $\dot{\psi}=\frac{d\psi}{dt}$; the angular velocity of rotation about the straight line Oz', the magnitude of which is $\dot{\vartheta}=\frac{d\vartheta}{dt}$; and the angular velocity of rotation about the longitudinal axis Ox, the magnitude of which is $\dot{\gamma}=\frac{d\gamma}{dt}$. With the help of Fig. 13 it is easy to find the projections of the angular velocity of rotation along the airplane axes:

$$
\left.\begin{aligned}
\omega_x &= \dot{\gamma} + \dot{\psi}\sin\vartheta, \\
\omega_y &= \dot{\psi}\cos\vartheta\cos\gamma + \dot{\vartheta}\sin\gamma, \\
\omega_z &= -\dot{\psi}\cos\vartheta\sin\gamma + \dot{\vartheta}\cos\gamma.
\end{aligned}\right\} \tag{I.1}
$$

From these formulas it is easy to derive the inverse formulas

$$
\left.\begin{aligned}
\frac{d\vartheta}{dt} &= \dot{\vartheta} = \omega_y \sin\gamma + \omega_z \cos\gamma, \\
\frac{d\psi}{dt} &= \dot{\psi} = \frac{\omega_y\cos\gamma - \omega_z\sin\gamma}{\cos\vartheta}, \\
\frac{d\gamma}{dt} &= \dot{\gamma} = \omega_x - \operatorname{tg}\vartheta\,(\omega_y\cos\gamma - \omega_z\sin\gamma).
\end{aligned}\right\} \tag{I.2}
$$

The velocity vector of the center of gravity, $\bar{V}$, will be oriented relative to the airplane axes by the angle of attack α and the angle of sideslip β in the following manner (see Fig. 14); the angle of sidslip β will be the angle between the velocity vector and the symmetry plane of the airplane, the angle of attack α

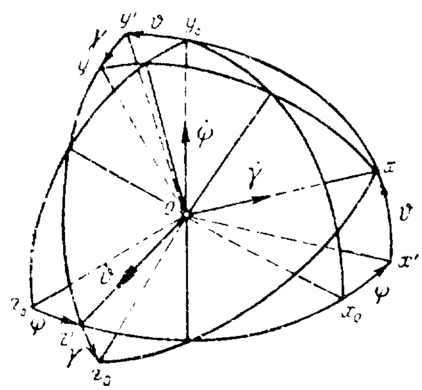

Fig. 13. Diagram of the angles of rotation of the coordinate exes during motion of the airplane.

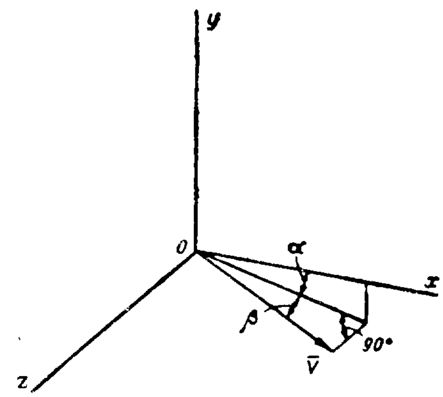

Fig. 14. Diagram of the angles of attack and sideslip.

will be the angle between the longitudinal airplane axis Ox and the projection of the velocity vector on the plane of symmetry. From Fig. 14 it is easy to find the projection of the velocity vector along the airplane axes:

$$
\left.\begin{aligned}
V_x &= V\cos\beta\cos\alpha, \\
V_y &= -V\cos\beta\sin\alpha, \\
V_z &= V\sin\beta,
\end{aligned}\right\} \tag{I.3}
$$

and also the inverse formulas

$$
\operatorname{tg}\beta = \frac{V_z}{\sqrt{V_x^2 + V_y^2}}, \quad \operatorname{tg}\alpha = -\frac{V_y}{V_x}, \quad V = \sqrt{V_x^2 + V_y^2 + V_z^2}. \tag{I.4}
$$

After these preliminary considerations, we can proceed to the derivation of the equations of perturbed lateral motion. We will assume, as has been done in this work, that the airplane coordinate axes are the principal axes of inertia. Then the moment equations for the axes Ox and Oy will be written in Eulerian form

$$J_x \frac{d\omega_x}{dt} + (J_z - J_y)\,\omega_y\omega_z = M_x,$$

$$J_y \frac{d\omega_y}{dt} + (J_x - J_z)\,\omega_x\omega_z = M_y.$$

Since the initial motion is assumed to be in the plane of symmetry, then in the initial motion

$$\beta = \omega_x = \omega_y = \gamma = 0.$$

In the derivation of linearized equations of perturbed lateral motion, the variations of these quantities are, in the first approximation, equal to the quantities themselves, which are infinitesimally small and of the first order. Considering this, Equations (I.1) and (I.3) will assume the form (disregarding small quantities of any order higher than the first):

$$\left.\begin{aligned} \omega_y &= \frac{d\psi}{dt}\cos\vartheta, \\ \omega_x &= \frac{d\gamma}{dt} + \frac{d\psi}{dt}\sin\vartheta = \frac{d\gamma}{dt} + \omega_y \operatorname{tg}\vartheta, \\ V_z &= V\beta. \end{aligned}\right\} \qquad (I.5)$$

In Euler's equations we disregard the products $\omega_x\omega_z$ and $\omega_x\omega_y$, as small quantities of the second order. Further, expanding the projections of the moments, M_x and M_y, as functions of β, ω_x, ω_y, $\delta_э$, δ_H' into a series and considering only terms of first order, we obtain

$$M_x = M_x^{\omega_x}\omega_x + M_x^{\omega_y}\omega_y + M_x^{\beta}\beta + M_x^{\delta э}\delta_э + M_x^{\delta H}\delta_H,$$
$$M_y = M_y^{\omega_x}\omega_x + M_y^{\omega_y}\omega_y + M_y^{\beta}\beta + M_y^{\delta э}\delta_э + M_y^{\delta H}\delta_H.$$

By substituting these expressions and the expression for ω_x (I.5) into Euler's moment equations, we obtain

$$\left.\begin{aligned} J_x\left(\frac{d\omega_x}{dt} + \frac{d\omega_y}{dt}\operatorname{tg}\vartheta\right) &= M_x^{\omega_x}\left(\frac{d\gamma}{dt} + \omega_y \operatorname{tg}\vartheta\right) + M_x^{\omega_y}\omega_y + M_x^{\beta}\beta + M_x^{\delta э}\delta_э + M_x^{\delta H}\delta_H, \\ J_y\frac{d\omega_y}{dt} &= M_y^{\omega_x}\left(\frac{d\gamma}{dt} + \omega_y \operatorname{tg}\vartheta\right) + M_y^{\omega_y}\omega_y + M_y^{\beta}\beta + M_y^{\delta э}\delta_э + M_y^{\delta H}\delta_H. \end{aligned}\right\} \qquad (I.6)$$

After grouping together the similar terms of this expression, it is easy to obtain two of the main Equations (4.1).

The equation of the projection of forces on the transverse airplane axis has the form:

$$m\left(\frac{dV_z}{dt} + \omega_x V_y - \omega_y V_x\right) = Z - mg\cos\vartheta\sin\gamma,$$

where Z is the projection of aerodynamic forces on the transverse axis O_z.

For infinitesimally small disturbances in the lateral motion,

$$V_z = V\beta, \quad \omega_x = \omega_y \operatorname{tg}\vartheta + \frac{d\gamma}{dt}, \qquad V_y = -V\sin\alpha,$$

$$V_x = V\cos\alpha, \qquad Z = Z^{\beta}\beta + Z^{\delta H}\delta_H^{*}.$$

By substituting the obtained expressions into the equation of force projections, we obtain

$$mV\frac{d\beta}{dt} - m\left(\omega_y \operatorname{tg}\vartheta + \frac{d\gamma}{dt}\right)V\sin\alpha - m\omega_y V\cos\alpha = Z^{\beta}\beta + Z^{\delta H}\delta_H - mg\cos\vartheta\gamma,$$

* We disregard the dependence of lateral force Z on ω_x, ω_y and $\delta_э$ because of their small influence on it. The quantity $Z^{\delta H}$ has to be considered because in contemporary airplanes it is equal to 30-50% of Z^{β}.

from which, after simple tranformations, it is easy to obtain the first of Equations (4.1), when we consider that $\vartheta = \theta + \alpha$ and, therefore,

$$\operatorname{tg}\vartheta \sin\alpha + \cos\alpha = \frac{\sin\vartheta\sin\alpha + \cos\vartheta\cos\alpha}{\cos\vartheta} = \frac{\cos\theta}{\cos\vartheta}.$$

Derivation of the equation of force projection on the lateral axis can also be accomplished in another way, which is interesting in itself, since in such a derivation the physical meaning of individual terms becomes clearer. For this we will find the projections of velocity vector V on the earth axes, by using Formulas (I.3) and the table of direction cosines:

$$\begin{gathered}
V_{x_0} = V_x \cos(x,x_0) + V_y \cos(y,x_0) + V_z \cos(z,x_0) = \\
= V\cos\alpha\cos\beta\cos\vartheta\cos\psi - V\sin\alpha\cos\beta\,[\sin\gamma\sin\psi - \cos\gamma\cos\psi\sin\vartheta] + \\
+ V\sin\beta\,[\cos\gamma\sin\psi + \sin\gamma\cos\psi\sin\vartheta], \\
V_{z_0} = V\cos(x,z_0) + V\cos(y,z_0) + V\cos(z,x_0) = \\
= -V\cos\alpha\cos\beta\cos\vartheta\sin\psi - V\sin\alpha\cos\beta\,[\sin\gamma\cos\psi + \cos\gamma\sin\psi\sin\vartheta] + \\
+ V\sin\beta\,[\cos\gamma\cos\psi - \sin\gamma\sin\psi\sin\vartheta].
\end{gathered}$$

On the other hand, the same projections can be obtained by projecting the velocity vector directly on the horizontal plane Ox_0z_0, and then resolving this projection along the axes Ox_0 and Oz_0. We obtain

$$V_{x_0} = V\cos\theta\cos\Pi, \qquad V_{z_0} = V\cos\theta\sin\Pi,$$

Where Π is the path angle, that is, the angle between axis Ox_0 and the projection of the velocity vector on the horizontal plane; in other words, it is the angle between axis Ox_0 and the tangent to the course on the earth.

Comparing the two groups of formulas it is easy to obtain

$$\begin{gathered}
\cos\theta\cos\Pi = \cos\alpha\cos\beta\cos\gamma\cos\psi - \sin\alpha\cos\beta\,[\sin\gamma\sin\psi - \cos\gamma\cos\psi\sin\vartheta] + \\
+ \sin\beta\,[\cos\gamma\sin\psi + \sin\gamma\cos\psi\sin\vartheta], \\
\cos\theta\sin\Pi = -\cos\alpha\cos\beta\cos\vartheta\sin\psi - \sin\alpha\cos\beta\,[\sin\gamma\cos\psi + \cos\gamma\sin\psi\sin\vartheta] + \\
+ \sin\beta\,[\cos\gamma\cos\psi - \sin\gamma\sin\psi\sin\vartheta].
\end{gathered}$$

From this we obtain the formulas

$$\left.\begin{aligned}
\cos\theta\cos(\psi + \Pi) &= \cos\alpha\cos\beta\cos\vartheta + \sin\alpha\cos\beta\cos\gamma\sin\vartheta + \sin\beta\sin\gamma\sin\vartheta, \\
\cos\theta\sin(\psi + \Pi) &= -\sin\alpha\cos\beta\sin\gamma + \sin\beta\cos\gamma.
\end{aligned}\right\} \quad (I.7)$$

With small lateral deviations, when the value of $\psi + \Pi$ is small, these formulas assume the form

$$\begin{gathered}
\cos\theta = \cos\alpha\cos\vartheta + \sin\alpha\sin\vartheta, \\
(\psi + \Pi)\cos\theta = \gamma\sin\alpha + \beta.
\end{gathered}$$

The first of these formulas gives us the well-known relation $\theta = \vartheta - \alpha$. The second allows us to obtain Π.

$$\Pi = -\psi + \frac{1}{\cos\theta}\beta - \frac{\sin\alpha}{\cos\theta}\gamma.$$

The velocity along the earth course equals the velocity vector projection on the horizontal plane

$$V_1 = V\cos\theta.$$

Therefore, the normal acceleration during motion along the course equals

$$W_n = V_1\frac{d\Pi}{dt} = V\cos\theta\left[-\frac{d\psi}{dt} + \frac{1}{\cos\theta}\frac{d\beta}{dt} - \frac{\sin\alpha}{\cos\theta}\frac{d\gamma}{dt}\right] = -V\cos\theta\frac{d\psi}{dt} + V\frac{d\beta}{dt} - V\sin\alpha\frac{d\gamma}{dt}.$$

Because $\frac{d\psi}{dt} = \frac{\omega_y}{\cos\vartheta}$ (Equation 1.5), then finally

$$W_n = -V\frac{\cos\theta}{\cos\vartheta}\omega_y + V\frac{d\beta}{dt} - V\sin\alpha\frac{d\gamma}{dt}.$$

From this, it is easy to obtain the lateral force equation

$$mV\left[\frac{d\beta}{dt} - \omega_y\frac{\cos\theta}{\cos\vartheta} - \sin\alpha\frac{d\gamma}{dt}\right] = Z^{\beta}\beta + Z^{\delta_H}\delta_H - mg\cos\vartheta\gamma.$$

APPENDIX II

FORMULAS FOR TRANSFORMING LATERAL ROTATIONAL DERIVATIVES TO THE NEW COORDINATE AXES

In this work the principal axes of inertia of the airplane are taken as the coordinate axes. At the same time, the static and rotational derivatives are usually calculated or determined experimentally in wind tunnels relative to the velocity axes. Therefore, in this appendix we present formulas for transforming the rotational derivatives for lateral motion in a rotation of the coordinate axes.

Let us denote the initial axes (Fig. 15) by x and y, and the coordinate axis rotated through the angle φ, by x' and y'. The projections of the vectors of angular velocity $\bar{\omega}$ and lateral force moment $\overline{M}$ on the coordinate axes x', y' will be

$$\left.\begin{aligned} M_x' &= M_x\cos\varphi + M_y\sin\varphi, & M_y' &= -M_x\sin\varphi + M_y\cos\varphi,\\ \omega_x' &= \omega_x\cos\varphi + \omega_y\sin\varphi, & \omega_y' &= -\omega_x\sin\varphi + \omega_y\cos\varphi. \end{aligned}\right\} \quad (II.1)$$

In the future, all quantities referred to the new axes will be distinguished by a prime (for example ω'_x). With these designations we will obtain the following transformation formulas for the rotational derivatives

$$\left.\begin{aligned} M_x'^{\omega_x'} &= M_x^{\omega_x}\cos^2\varphi + (M_x^{\omega_y} + M_y^{\omega_x})\cos\varphi\sin\varphi + M_y^{\omega_y}\sin^2\varphi,\\ M_x'^{\omega_y'} &= M_x^{\omega_y}\cos^2\varphi + (M_y^{\omega_y} - M_x^{\omega_x})\cos\varphi\sin\varphi - M_y^{\omega_x}\sin^2\varphi,\\ M_y'^{\omega_x'} &= M_y^{\omega_x}\cos^2\varphi + (M_y^{\omega_y} - M_x^{\omega_x})\cos\varphi\sin\varphi - M_x^{\omega_y}\sin^2\varphi,\\ M_y'^{\omega_y'} &= M_y^{\omega_y}\cos^2\varphi - (M_x^{\omega_y} + M_y^{\omega_x})\cos\varphi\sin\varphi + M_x^{\omega_x}\sin^2\varphi. \end{aligned}\right| \quad (II.2)$$

Let us derive the first two formulas. In order to do that, let us note that the projection on the axis x' of that part of the moment which results from the angular velocity, equals: on one hand (we are considering only the part of the moment which is caused by angular velocity of rotation)

$$M_x' = M_x'^{\omega_x'}\omega_x' + M_x'^{\omega_y'}\omega_y' = M_x'^{\omega_x'}(\omega_x\cos\varphi + \omega_y\sin\varphi) + M_x'^{\omega_y'}(-\omega_x\sin\varphi + \omega_y\cos\varphi),$$

on the other hand

$$M_x' = M_x\cos\varphi + M_y\sin\varphi = (M_x^{\omega_x}\omega_x + M_x^{\omega_y}\omega_y)\cos\varphi + (M_y^{\omega_x}\omega_x + M_y^{\omega_y}\omega_y)\sin\varphi.$$

Comparing the coefficients of ω_x and ω_y in both expressions, we obtain

$$M_x'^{\omega_x'} \cos\varphi - M_x'^{\omega_y'} \sin\varphi = M_x^{\omega_x} \cos\varphi + M_y^{\omega_x} \sin\varphi,$$

$$M_x'^{\omega_x'} \sin\varphi + M_x'^{\omega_y'} \cos\varphi = M_x^{\omega_y} \cos\varphi + M_y^{\omega_y} \sin\varphi.$$

Solving these equations for $M_x'^{\omega x}$ and $M_x'^{\omega' y}$, it is easy to obtain the first two formulas of (II.2); in an analogous fashion we obtain the two second formulas by comparing two expressions for M_y'. The dimensionless coefficients involving the rotational derivatives can also be transformed by the same formulas.

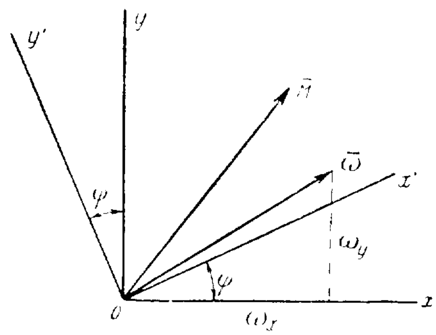

Fig. 15.

Transformations according to Formulas (II.2) can be simplified, if it is taken into account that the transformation to the new axes through formulas (II.2) has the two invariants

$$\left.\begin{aligned} J_1 &= M_x'^{\omega_x'} + M_y'^{\omega_y'} = M_x^{\omega_x} + M_y^{\omega_y}, \\ J_2 &= M_x'^{\omega_x'} M_y'^{\omega_y'} - M_x'^{\omega_y'} M_y'^{\omega_x'} = M_x^{\omega_x} M_y^{\omega_y} - M_x^{\omega_y} M_y^{\omega_x}. \end{aligned}\right| \quad \text{(II.3)}$$

In the transformation Formulas (II.2) for small angles φ, when $\sin\varphi$ is small in comparison with $\cos\varphi$, the principal variable terms are the second terms with the multipliers $\cos\varphi \sin\varphi$. Since $M_x^{\omega x}$ is usually considerably larger than $M_y^{\omega y}$ and $M_x^{\omega y}$, and $M_y^{\omega x}$ is even smaller, then in the transformation the value of $M_y^{\omega x}$; changes more than the others; depending on the value of angle φ it can even change its sign.

The result of converting the rotational derivatives cited in section 5 to new axes, inclined to the old axes by the angle $\varphi = \pm 15°$, is given in Table VII.

TABLE VII

φ	−15°	0	+15°
$\bar{m}_x^{\omega x}$	−0,342	−0,390	−0,413
$\bar{m}_x^{\omega y}$	−0,120	−0,125	−0,110
$\bar{m}_y^{\omega x}$	−0,059	−0,020	0,039
$\bar{m}_y^{\omega y}$	−0,243	−0,195	−0,172

It can be seen clearly from this table that the quantity $m_y^{\omega_x}$ undergoes the greatest change. This quantity even changes sign.

The remaining quantities are changed relatively little. We note that it is just this coefficient $m_y^{\omega_x}$ which determines the characteristics of link V, and, in particular, the value of the reference frequency ω_5 (see Section 5). Consequently, in contemporary subsonic airplanes, and in supersonic airplanes even more, the angle of inclination of the axis of inertia has a large effect on the stability characteristics.

When converting from velocity axes to principal axes, the angle $\varphi = -\alpha$, where α is the angle of attack, that is, the angle between the flight velocity vector and the principal axis of inertia.

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